

# BASES OF SCHURIAN ANTISYMMETRIC COHERENT CONFIGURATIONS AND ISOMORPHISM TEST FOR SCHURIAN TOURNAMENTS

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ABSTRACT. It is known that for any permutation group  $G$  of odd order one can find a subset of the permuted set whose stabilizer in  $G$  is trivial, and if  $G$  is primitive, then also a base of size at most 3. Both of these results are generalized to the coherent configuration of  $G$  (that is in this case a schurian antisymmetric coherent configuration). This enables us to construct a polynomial-time algorithm for recognizing and isomorphism testing of schurian tournaments (i.e. arc colored tournaments the coherent configurations of which are schurian).

## 1. INTRODUCTION

Let  $\mathcal{X}$  be a coherent configuration (as for the background of coherent configurations we refer to Section 2 and [10]). A *base* of  $\mathcal{X}$  is a point set  $\Delta$  such that the smallest fission of  $\mathcal{X}$  in which all points of  $\Delta$  are fibers, is the complete coherent configuration.<sup>1</sup> The minimal size of  $\Delta$  is called the *base number* of  $\mathcal{X}$  and is denoted by  $b(\mathcal{X})$ . It is easily seen that  $0 \leq b(\mathcal{X}) \leq n-1$  where  $n$  is the degree of  $\mathcal{X}$ . Besides, given a permutation group  $G$  denote by  $b(G)$  the smallest size of a base of  $G$ .<sup>2</sup> Then

$$(1) \quad b(G) \leq b(\text{Inv}(G))$$

where  $\text{Inv}(G)$  is the coherent configuration associated with  $G$ . A weaker upper bound for  $b(G)$  enables us to estimate the maximal order of uniprimitive group as it was done in [2]. It also follows from Theorem 0.2 of that paper that the base number of a nontrivial primitive coherent configuration of degree  $n$  is less than  $4\sqrt{n} \log n$ .

The equality in (1) is obviously attained when the group  $G$  is trivial or symmetric. A nontrivial example of the equality was found in [8] for  $G$  being the automorphism group of a cyclotomic scheme over finite field. On the other hand, in general inequality (1) is strict even for solvable groups: if  $G$  is a solvable 2-transitive group of degree  $n$ , then  $b(G) \leq 4$  by [19], but in this case  $\text{Inv}(G)$  is trivial, and hence  $b(\text{Inv}(G)) = n-1$ . In contrast to this example we prove here the following theorem.

**Theorem 1.1.** *Let  $G$  be a primitive permutation group of odd order. Then the base number of the coherent configuration  $\text{Inv}(G)$  is at most 3.*

As an immediate consequence of inequality (1) and Theorem 1.1 we deduce that the base number of a primitive permutation group of odd order is at most 3 (this

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<sup>1</sup>In survey [4] the name EP-base was used.

<sup>2</sup>A base of a permutation group is a set of permuted points whose pointwise stabilizer is trivial.

result have been earlier proved in [13]). In proving Theorem 1.1 we also get a generalization of the Gluck theorem that in any odd order permutation group the stabilizer of some subset of the permuted set is trivial [14]. Namely, a *generalized base* of a coherent configuration  $\mathcal{X}$  with the point set  $\Omega$  is a set  $\Pi \subset 2^\Omega$  such that the smallest fission of  $\mathcal{X}$  in which any element of  $\Pi$  is a union of some fibers of  $\mathcal{X}$ , is the complete coherent configuration. The minimal size of the set  $\Pi$  is called the *generalized base number* of  $\mathcal{X}$  and is denoted by  $gb(\mathcal{X})$ . Again, it is easily seen that

$$(2) \quad gb(G) \leq gb(\text{Inv}(G))$$

where  $gb(G)$  is the minimal size of the set  $\Pi$  for which the intersection of all  $G_{\{\Delta\}}$  with  $\Delta \in \Pi$  is trivial.

**Theorem 1.2.** *Let  $G$  be a permutation group of odd order. Then the generalized base number of the coherent configuration  $\text{Inv}(G)$  is at most 1.*

A coherent configuration  $\mathcal{X}$  is called *schurian* if there exists a permutation group  $G$  such that  $\mathcal{X} = \text{Inv}(G)$ . Over all coherent configurations schurian ones are relatively rare in occurrence. For example for infinitely many positive integers  $n$  there are exponentially many antisymmetric coherent configuration of rank 3 and degree  $n$ ; on the other hand, such a configuration is schurian if and only if it arises from the Payley tournament on  $n$  vertices. In this paper we apply Theorem 1.1 to get the following result.

**Theorem 1.3.** *Given an antisymmetric coherent configuration  $\mathcal{X}$  on  $n$  points one can test in time  $n^{O(1)}$  whether  $\mathcal{X}$  is schurian, and (if so) find the group  $\text{Aut}(\mathcal{X})$ .*

Antisymmetric coherent configurations are closely related with tournaments (we recall that tournament is a directed graph in which any two distinct vertices are joined by a unique arc). Indeed, one can easily see that if  $(\Omega, S)$  is such a configuration and  $A$  is a maximal subset of  $S$  such that  $A \cap A^* = \emptyset$ , then  $(\Omega, A)$  is a tournament. Conversely, the coherent configuration obtained from an arc colored tournament  $T$  by means of the Weisfeiler-Leman algorithm<sup>3</sup> is antisymmetric. When this configuration is schurian, we say that the tournament  $T$  is *schurian*. In particular, this is always the case when the color classes of arcs are the orbits of the group  $\text{Aut}(T)$  (acting on the pairs of vertices).

Let us turn to the tournament isomorphism problem. It is a special case of the Graph Isomorphism Problem that consists in finding an efficient algorithm to test whether or not two given (arc colored) tournaments are isomorphic. At present, the best result here is the algorithm from [3] testing the isomorphism of two  $n$ -vertex tournaments in time  $n^{O(\log n)}$  (see also [1]). In this paper we prove the following result.

**Theorem 1.4.** *Let  $\mathcal{T}_n$  be the class of all schurian tournaments on  $n$  vertices. Then the following problems can be solved in time  $n^{O(1)}$ :*

- (1) *given a tournament  $T$  on  $n$  vertices, test whether  $T \in \mathcal{T}_n$ ,*
- (2) *given a tournament  $T \in \mathcal{T}_n$  find the group  $\text{Aut}(T)$ ,*
- (3) *given tournaments  $T_1, T_2 \in \mathcal{T}_n$  find a set  $\text{Iso}(T_1, T_2)$ .*

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<sup>3</sup>This algorithm was given in detail in [23]; see also Subsection 2.8.

The proof of Theorem 1.4 is reduced to Theorem 1.3). In the special case when  $\mathcal{X} = \text{Inv}(G)$  for an odd order group  $G$ , the group  $\text{Aut}(\mathcal{X})$  is by definition the 2-closure of  $G$ . Thus our algorithm, in particular, constructs in polynomial time the 2-closure of any odd order permutation group, that generalizes the main result in [7].

For the reader convenience we collect the basic facts on coherent configurations, their bases and linear primitive solvable groups in Sections 2, 3 and 4 respectively. In Sections 5, 6 and 7 we prove some upper bounds for the numbers  $b(\mathcal{X})$  and  $gb(\mathcal{X})$  when  $\mathcal{X}$  is the wreath product or the exponentiation of coherent configurations. In Section 8 we give a sufficient condition for a coherent configuration  $\text{Inv}(G)$  with transitive  $G$  to have a base of size at most 2. This condition is used in Section 9 where we prove that the equality in (1) attained when  $G$  is an affine linear group with irreducible zero stabilizer (Theorem 9.1). Finally, the proofs of Theorems 1.1, 1.2 and 1.4 are given in Section 10.

**Notation.** Throughout the paper  $\Omega$  denotes a finite set. The diagonal of the Cartesian product  $\Omega^2$  is denoted by  $1_\Omega$ .

For  $r \subset \Omega^2$  set  $r^* = \{(\beta, \alpha) : (\alpha, \beta) \in r\}$ . For  $\Gamma, \Delta \subset \Omega$  set  $r_{\Gamma, \Delta} = r \cap (\Gamma \times \Delta)$  and  $r_\Gamma = r_{\Gamma, \Gamma}$ .

For any  $\alpha \in \Omega$  set  $1_\alpha = 1_{\{\alpha\}}$  and  $\alpha r = \{\beta \in \Omega : (\alpha, \beta) \in r\}$ .

For  $r, s \subset \Omega^2$  set  $r \cdot s = \{(\alpha, \beta) \in \Omega^2 : (\alpha, \gamma) \in r, (\gamma, \beta) \in s \text{ for some } \gamma \in \Omega\}$ , and set  $r \otimes s = \{(\alpha, \beta) \in \Omega^2 \times \Omega^2 : (\alpha_1, \beta_1) \in r \text{ and } (\alpha_2, \beta_2) \in s\}$ .

For  $S \in 2^{\Omega^2}$  denote by  $S^\cup$  the set of all unions of the elements of  $S$ , and set  $S^* = \{s^* : s \in S\}$  and  $\alpha S = \cup_{s \in S} \alpha s$ . For  $T \in 2^{\Omega^2}$  set  $S \cdot T = \{s \cdot t : s \in S, t \in T\}$ .

For a permutation  $g$  set  $\text{fix}(g)$  to be the number of points that  $g$  leaves fixed. For a set  $K$  of permutations set  $\text{fix}(K) = \max_g \text{fix}(g)$  and  $\text{Fix}(K) = \sum_g \text{fix}(g)$  where  $g$  runs over the set  $K^\# = K \setminus \{1\}$ .

## 2. COHERENT CONFIGURATIONS

Unfortunately up to now there is no commonly used terminology and notations in the coherent configuration theory. In what follows we use a mix from [10] and [24]. All the facts presented below can be found in one of these sources.

**2.1. Definitions.** A pair  $\mathcal{X} = (\Omega, S)$  where  $\Omega$  is a finite set and  $S$  a partition of  $\Omega^2$ , is called a *coherent configuration* on  $\Omega$  if  $1_\Omega \in S^\cup$ ,  $S^* = S$ , and given  $r, s, t \in S$ , the number

$$c_{rs}^t = |\alpha r \cap \beta s^*|$$

does not depend on the choice of  $(\alpha, \beta) \in t$ . The elements of  $\Omega$ ,  $S$ ,  $S^\cup$  and the numbers  $c_{rs}^t$  are called the *points*, the *basic relations*, the *relations* and the *intersection numbers* of  $\mathcal{X}$ , respectively. For the intersection numbers the following equalities hold:

$$(3) \quad c_{r^*s^*}^{t^*} = c_{sr}^t \quad \text{and} \quad |t|c_{rs}^{t^*} = |r|c_{st}^{r^*} = |s|c_{tr}^{s^*}, \quad r, s, t \in S.$$

The numbers  $|\Omega|$  and  $|S|$  are called the *degree* and the *rank* of  $\mathcal{X}$ . A unique basic relation containing a pair  $(\alpha, \beta) \in \Omega^2$  is denoted by  $r_{\mathcal{X}}(\alpha, \beta)$  or  $r(\alpha, \beta)$ . The set of basic relations contained in  $r \cdot s$  with  $r, s \in S^\cup$  is denoted by  $rs$ .

**2.2. Fibers and homogeneity.** The point set  $\Omega$  is a disjoint union of *fibers* which are the elements of the set

$$\Phi(\mathcal{X}) = \{\Gamma \subset \Omega : 1_\Gamma \in S\}$$

Given a union  $\Delta$  of fibers denote by  $S_\Gamma$  the set of all nonempty relations  $r_\Gamma$  with  $r \in S$ . Then  $\mathcal{X}_\Gamma = (\Gamma, S_\Gamma)$  is a coherent configuration, called the *restriction* of  $\mathcal{X}$  to  $\Gamma$ .

For any basic relation  $r \in S$  there exist uniquely determined fibers  $\Gamma$  and  $\Delta$  such that  $r \subset \Gamma \times \Delta$ . The number  $|\gamma r| = c_{ss^*}^t$  with  $t = 1_\Gamma$ , does not depend on  $\gamma \in \Gamma$ . It is called the *valency* of  $r$  and denoted  $n_r$ . The maximum of all valences is denoted by  $n_{max}$ .

The coherent configuration  $\mathcal{X}$  is called *homogeneous* or a *scheme* if  $1_\Omega \in S$ . In this case  $n_r = n_{r^*}$  and  $|r| = nn_r$  for all  $r \in S$  where  $n = |\Omega|$ . Thus equalities in (3) may be rewritten as follows:

$$(4) \quad c_{r^*s^*}^{t^*} = c_{sr}^t \quad \text{and} \quad n_t c_{rs}^{t^*} = n_r c_{st}^{r^*} = n_s c_{tr}^{s^*}.$$

**2.3. Equivalence relations.** Let us define the *support* of a relation  $r \subset \Omega^2$  to be the minimal set  $\Gamma \subset \Omega$  such that  $r \subset \Gamma^2$ . Saying that  $e \in S^\cup$  is an equivalence relation we mean that  $e$  is an equivalence relation on its support; the set of classes of  $e$  is denoted by  $\Omega/e$ . According to [11, Subsection 3.2] any such  $e$  is a union of uniform equivalence relations<sup>4</sup> belonging to  $S^\cup$  and having pairwise disjoint supports. This implies the following statement to be used in the proof of Corollary 5.2.

**Lemma 2.1.** *Let  $\mathcal{X}$  be a coherent configuration on  $\Omega$ ,  $e \in S^\cup$  is an equivalence relation and  $I \subset \Omega/e$ . Suppose that no two classes of  $e$ , one is in  $I$  and another one not in  $I$ , have the same size. Then the union of all elements of  $I$  belongs to the set  $\Phi(\mathcal{X})^\cup$ .* ■

Any coherent configuration has *trivial* equivalence relations in  $S^\cup$ :  $1_\Omega$  and  $\Omega^2$ . A homogeneous coherent configuration is called *primitive* if there are no other equivalence relations in  $S^\cup$ ; otherwise it is called *imprimitive*.

Let  $e \in S^\cup$  be an equivalence relation. Then given  $\Gamma \in \Omega/e$  one can construct the *restriction* of  $\mathcal{X}$  to  $\Gamma$  that is the coherent configuration

$$\mathcal{X}_\Gamma = (\Gamma, S_\Gamma)$$

with  $S_\Gamma$  as in Subsection 2.2. The *quotient* of  $\mathcal{X}$  modulo  $e$  is defined to be the coherent configuration

$$\mathcal{X}_{\Omega/e} = (\Omega/e, S_{\Omega/e})$$

where  $S_{\Omega/e}$  is the set of all nonempty relations of the form  $\{(\Gamma, \Delta) : s_{\Gamma, \Delta} \neq \emptyset\}$  with  $s \in S$ .

**2.4. Fissions and fusions.** There is a natural partial order  $\leq$  on the set of all coherent configurations on the set  $\Omega$ . Namely, given two coherent configurations  $\mathcal{X} = (\Omega, S)$  and  $\mathcal{X}' = (\Omega, S')$  we set

$$\mathcal{X} \leq \mathcal{X}' \Leftrightarrow S^\cup \subset (S')^\cup.$$

In this case  $\mathcal{X}$  and  $\mathcal{X}'$  are called respectively a *fusion* of  $\mathcal{X}'$  and a *fission* of  $\mathcal{X}$ . This order is preserved under taking the restriction to a set and the quotient modulo

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<sup>4</sup>An equivalence relation is uniform if all its classes have the same size.

an equivalence. The minimal and maximal elements with respect to that order are the *trivial* and the *complete* coherent configurations on  $\Omega$ : the basis relations of the former one are the reflexive relation  $1_\Omega$  and (if  $n > 1$ ) its complement in  $\Omega^2$ , whereas the relations of the latter one are all binary relations on  $\Omega$ .

Given two coherent configurations  $\mathcal{X}_1$  and  $\mathcal{X}_2$  on  $\Omega$  there is a uniquely determined coherent configuration  $\mathcal{X}_1 \cap \mathcal{X}_2$  also on  $\Omega$ , the relation set of which is  $(S_1)^\cup \cap (S_2)^\cup$  where  $S_i$  is the set of basis relations of  $\mathcal{X}_i$ ,  $i = 1, 2$ . This enables us to define the smallest fission of a coherent configuration  $\mathcal{X}$  on  $\Omega$  containing a given set  $\mathcal{S}$  of binary relations on  $\Omega$  as follows:

$$\text{Fis}(\mathcal{X}, \mathcal{S}) = \bigcap_{\mathcal{Y}: \mathcal{S} \subset T^\cup} \mathcal{Y}$$

where  $\mathcal{Y} = (\Omega, T)$  is a coherent configuration. In what follows we will omit  $\mathcal{X}$  when it is the trivial coherent configuration. Besides, for  $\Pi \subset 2^\Omega$  and  $\Gamma \subset \Omega$  we define respectively the  $\Pi$ -fission and  $\Gamma$ -fission of  $\mathcal{X}$  by

$$\text{Fis}(\mathcal{X}, \Pi) = \text{Fis}(\mathcal{X}, \mathcal{S}_\Pi) \quad \text{and} \quad \text{Fis}(\mathcal{X}, \Gamma) = \text{Fis}(\mathcal{X}, \Pi_\Gamma)$$

where  $\mathcal{S}_\Pi = \{1_\Delta : \Delta \in \Pi\}$  and  $\Pi_\Gamma = \{\{\gamma\} : \gamma \in \Gamma\}$ . Sometimes we will also write  $\mathcal{X}_{\alpha, \beta, \dots}$  instead of  $\text{Fis}(\mathcal{X}, \{\alpha, \beta, \dots\})$ . One can see that any set in  $\Pi^\cup$  is a union of fibers of the  $\Pi$ -fission of  $\mathcal{X}$ . The following lemma immediately follows from the definitions.

**Lemma 2.2.** *Let  $\mathcal{X} = (\Omega, S)$  be a coherent configuration and  $\alpha \in \Omega$ . Then for all  $r, s, t \in S$  we have*

$$\alpha r \in (\Phi_\alpha)^\cup \quad \text{and} \quad t_{\alpha r, \alpha s} \in (S_\alpha)^\cup$$

where  $\Phi_\alpha$  and  $S_\alpha$  are the sets of fibers and basis relations of the coherent configuration  $\mathcal{X}_\alpha$ . Moreover,  $|\beta t_{\alpha r, \alpha s}| = c_{rt}^s$  for all  $\beta \in \alpha r$ . ■

**2.5. Isomorphisms and schurity.** Two coherent configurations  $\mathcal{X}_1$  and  $\mathcal{X}_2$  are called *isomorphic* if there exists a bijection between their point sets that induces a bijection between their sets of basic relations. Such a bijection is called an *isomorphism* between  $\mathcal{X}_1$  and  $\mathcal{X}_2$ ; the set of all of them is denoted by  $\text{Iso}(\mathcal{X}_1, \mathcal{X}_2)$ .

The group of all isomorphisms of a coherent configuration  $\mathcal{X} = (\Omega, S)$  to itself contains a normal subgroup

$$\text{Aut}(\mathcal{X}) = \{f \in \text{Sym}(\Omega) : s^f = s, s \in S\}$$

called the *automorphism group* of  $\mathcal{X}$  where  $s^f = \{(\alpha^f, \beta^f) : (\alpha, \beta) \in s\}$ . Conversely, let  $G$  be a permutation group on  $\Omega$  and  $S$  the set of orbits of the componentwise action of  $G$  on  $\Omega^2$ . Then  $\text{Inv}(G) := (\Omega, S)$  is a coherent configuration; it is called the *coherent configuration of  $G$* . This coherent configuration is homogeneous if and only if the group  $G$  is transitive. One can also see that

$$\mathcal{X} \leq \mathcal{X}' \Rightarrow \text{Aut}(\mathcal{X}) \geq \text{Aut}(\mathcal{X}') \quad \text{and} \quad G \leq G' \Rightarrow \text{Inv}(G) \geq \text{Inv}(G').$$

A coherent configuration  $\mathcal{X}$  is called *schurian* if  $\mathcal{X} = \text{Inv}(G)$  for some permutation group  $G$ . In this case the group  $G$  can be always replaced by  $\text{Aut}(\mathcal{X})$ . Moreover, the schurity of  $\mathcal{X}$  implies the schurity of all its restrictions and quotients. An important example of a schurian scheme is a *cyclotomic scheme* over a finite field  $\mathbb{F}$ ; in this case  $G$  is an affine subgroup of  $\text{AGL}(1, \mathbb{F})$ .<sup>5</sup> In this paper we also

<sup>5</sup>In what follows saying that  $G$  is an affine (sub)group we mean that  $G$  contains all the translations of the underlying linear space.

deal with the scheme of a primitive solvable group. The structure of such a group is given in the following statement proved in [21, Section 4].

**Theorem 2.3.** *Let  $G \leq \text{Sym}(\Omega)$  be a primitive solvable permutation group. Then  $|\Omega| = p^d$  for a prime  $p$  and integer  $d \geq 1$ . Moreover the set  $\Omega$  can be identified with a linear space of dimension  $d$  over field of order  $p$  so that*

$$G \leq \text{AGL}(d, p) \quad \text{and} \quad K \leq \text{GL}(d, p)$$

where  $K$  is the stabilizer of zero point in  $G$ ; the group  $G$  is affine and the group  $K$  is irreducible.  $\blacksquare$

**2.6. Algebraic isomorphisms.** Let  $\mathcal{X} = (\Omega, S)$  and  $\mathcal{X}' = (\Omega', S')$  be coherent configurations. A bijection  $\varphi : S \rightarrow S'$ ,  $r \mapsto r'$  is called an *algebraic isomorphism* from  $\mathcal{X}$  to  $\mathcal{X}'$  if

$$(5) \quad c_{rs}^t = c_{r's'}^{t'}, \quad r, s, t \in S;$$

we say that  $\mathcal{X}$  and  $\mathcal{X}'$  are *algebraically isomorphic*. In this case they have the same degree and rank. Moreover,  $\varphi$  induces a bijection from  $S^\cup$  onto  $(S')^\cup$  such that

$$(r \cup s)^\varphi = r^\varphi \cup s^\varphi, \quad r, s \in S.$$

This bijection preserves reflexive and equivalence relations. In particular, we can define a bijection from  $\Phi(\mathcal{X})^\cup$  onto  $\Phi(\mathcal{X}')^\cup$  so that  $(1_\Gamma)^\varphi = 1_{\Gamma^\varphi}$ . Finally, given a set  $\Gamma \in \Phi(\mathcal{X})^\cup$  and an equivalence relation  $e \in S^\cup$  we have the induced algebraic isomorphisms

$$\varphi_\Gamma : \mathcal{X}_\Gamma \rightarrow \mathcal{X}'_{\Gamma'} \quad \text{and} \quad \varphi_{\Omega/e} : \mathcal{X}_{\Omega/e} \rightarrow \mathcal{X}'_{\Omega'/e'}$$

where  $\Gamma' = \Gamma^\varphi$  and  $e' = e^\varphi$ .

Any isomorphism  $f \in \text{Iso}(\mathcal{X}, \mathcal{X}')$  induces an algebraic isomorphism  $r \mapsto r^f$  from  $\mathcal{X}$  to  $\mathcal{X}'$ . The set of all isomorphisms inducing the algebraic isomorphism  $\varphi$  is denoted by  $\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ . Clearly,

$$\text{Iso}(\mathcal{X}, \mathcal{X}, \text{id}_S) = \text{Aut}(\mathcal{X})$$

where  $\text{id}_S$  is the identity on  $S$ . Let us give another example of algebraic isomorphism. Suppose that the scheme  $\mathcal{X}$  is imprimitive and  $e \in S^\cup$  an equivalence relation. Then given any two sets  $\Gamma, \Gamma' \in \Omega/e$  the mapping

$$(6) \quad \varphi_{\Gamma, \Gamma'} : \mathcal{X}_\Gamma \rightarrow \mathcal{X}_{\Gamma'}, \quad s_\Gamma \rightarrow s_{\Gamma'}$$

is an algebraic isomorphism (here  $s$  runs over the set of all basis relations of  $\mathcal{X}$  that are contained in  $e$ ).

**2.7. Antisymmetric and 1-regular coherent configurations.** A coherent configuration  $\mathcal{X}$  is called *antisymmetric* if

$$s \in S \quad \text{and} \quad s = s^* \quad \Rightarrow \quad s \subset 1_\Omega,$$

or equivalently if the cardinality of any basis relation of  $\mathcal{X}$  is an odd number. The latter condition implies that the valences of  $\mathcal{X}$  are odd and that the coherent configuration  $\text{Inv}(G)$  is antisymmetric if and only if  $G$  is the group of odd order. One can prove that the class of antisymmetric coherent configurations is closed with respect to taking fissions, restrictions and quotients. In particular, the automorphism group of antisymmetric coherent configuration has odd order.

A coherent configuration  $\mathcal{X}$  is called *1-regular* if it has a *regular* point; by definition a point  $\alpha \in \Omega$  is regular in  $\mathcal{X}$ , if

$$(7) \quad r \in S \Rightarrow |\alpha r| \leq 1.$$

The set  $\Gamma$  of all regular points is a union of fibers and any basic relation of the coherent configuration  $\mathcal{X}_\Gamma$  has valency 1. When  $\Omega = \Gamma$ , the coherent configuration  $\mathcal{X}$  is called *semiregular*, and *regular* in homogeneous case. Thus regular schemes are exactly *thin schemes* in the sense of [24]. One can also define such a scheme by the condition that any basis relation  $r$  of it is *thin*, i.e. that

$$|\alpha r| \leq 1 \quad \text{and} \quad |\alpha r^*| \leq 1$$

for all  $\alpha \in \Omega$ . We note that the set of all thin relations on the same set is closed with respect to  $*$  and  $\cdot$ .

**2.8. The Weisfeiler-Leman algorithm.** From the algorithmic point of view a coherent configuration  $\mathcal{X}$  on  $n$  points is given by the set  $S$  of its basis relations. In this representation one can check in time  $n^{O(1)}$  whether  $\mathcal{X}$  is homogeneous or imprimitive. Moreover, within the same time one can list the fibers of  $\mathcal{X}$ , and find a nontrivial equivalence relation  $e \in S^\cup$  (if it exists) as well as the quotient of  $\mathcal{X}$  modulo  $e$ .

The well-known Weisfeiler-Leman algorithm is described in detail in [23, Section B]. The input of it is a set  $\mathcal{S}$  of binary relations on a set  $\Omega$ , and the output is the coherent configuration  $\mathcal{X} = \text{Fis}(\mathcal{S})$ . The running time of the algorithm is polynomial in sizes of  $\mathcal{S}$  and  $\Omega$ . The canonical version of the Weisfeiler-Leman algorithm have been studied in Section M of the above book (under the name simultaneous stabilization), where in fact the following statement was proved.

**Theorem 2.4.** *Let  $\mathcal{S}_i$  be a set of  $m$  binary relations on a set of size  $n$ ,  $i = 1, 2$ . Then given a bijection  $\psi : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  one can check in time  $mn^{O(1)}$  whether or not there exists an algebraic isomorphism  $\varphi : \text{Fis}(\mathcal{S}_1) \rightarrow \text{Fis}(\mathcal{S}_2)$  such that  $\varphi|_{\mathcal{S}_1} = \psi$ . Moreover, if  $\varphi$  does exist, it can be found within the same time.* ■

### 3. BASES OF A COHERENT CONFIGURATION

**3.1. Generalized base.** A set  $\Pi \subset 2^\Omega$  is called a *generalized base* of a coherent configuration  $\mathcal{X}$  if the  $\Pi$ -fission of it is complete. When  $\Pi$  consists of singletons, we call it the *base* of  $\mathcal{X}$  and identify it with the corresponding subset of  $\Omega$ . It is easily seen that  $\Pi$  is a generalized base of any fission of  $\mathcal{X}$ , and that any  $\Pi' \subset 2^\Omega$  that contains  $\Pi$  is a generalized base of  $\mathcal{X}$ . It is also clear that replacing some elements of  $\Pi$  by their complements in  $\Omega$  produces a generalized base of  $\mathcal{X}$ . The following simple statement will be used in Section 7.

**Lemma 3.1.** *Let  $\mathcal{X}$  be a coherent configuration on  $\Omega$ ,  $\Pi$  a generalized base of  $\mathcal{X}$  and  $\alpha \in \Omega$ . Set  $\mathcal{X}' = (\mathcal{X}_\alpha)_{\Omega'}$  and  $\Pi' = \{\Gamma' : \Gamma \in \Pi\}$  where  $\Gamma' = \Gamma \setminus \{\alpha\}$  for all  $\Gamma \subset \Omega$ . Then  $\Pi'$  is a generalized base of  $\mathcal{X}'$ .*

**Proof.** Denote by  $\mathcal{Y}$  the direct sum of the one-point coherent configuration on  $\{\alpha\}$  and  $\Pi'$ -fission of  $\mathcal{X}'$ , i.e. the smallest coherent configuration on  $\Omega$  such that

$$\{\alpha\} \in \Phi(\mathcal{Y}) \quad \text{and} \quad \mathcal{Y}_{\Omega'} = \text{Fis}(\mathcal{X}', \Pi')$$

Then obviously  $\mathcal{Y} \geq \mathcal{X}_\alpha$  and  $\Pi \subset \Phi(\mathcal{Y})^\cup$ . Therefore

$$\mathcal{Y} \geq \text{Fis}(X_\alpha, \Pi) \geq \text{Fis}(\mathcal{X}, \Pi).$$

Since  $\Pi$  is a generalized base of  $\mathcal{X}$ , it follows that  $\text{Fis}(\mathcal{X}, \Pi)$ , and hence  $\mathcal{Y}$ , is a complete coherent configuration. This implies that so is  $\mathcal{Y}_{\Omega'} = \text{Fis}(\mathcal{X}', \Pi')$ . Thus  $\Pi'$  is a generalized base of  $\mathcal{X}'$ .  $\blacksquare$

**3.2. Generalized base number.** The smallest cardinality  $gb(\mathcal{X})$  (resp. by  $b(\mathcal{X})$ ) of a generalized base (resp. of a base) of the coherent configuration  $\mathcal{X}$  is called the *generalized base number* (resp. the *base number*) of  $\mathcal{X}$ . Obviously,

$$(8) \quad gb(\mathcal{X}) \leq b(\mathcal{X}).$$

Since any fiber of  $\mathcal{X}$  is a union of fibers in any its fission, we also have

$$(9) \quad gb(\mathcal{X}) \leq \max_{\Gamma \in \Phi} gb(\mathcal{X}_\Gamma) \quad \text{and} \quad b(\mathcal{X}) \leq \sum_{\Gamma \in \Phi} b(\mathcal{X}_\Gamma)$$

where  $\Phi = \Phi(\mathcal{X})$ . Moreover, from the remark made in Subsection 3.1 it immediately follows that

$$\mathcal{X}' \geq \mathcal{X} \quad \Rightarrow \quad gb(\mathcal{X}') \leq gb(\mathcal{X}) \quad \text{and} \quad b(\mathcal{X}') \leq b(\mathcal{X}).$$

It was observed in [8] that any regular point of a coherent configuration forms a base of it. Thus  $b(\mathcal{X}) \leq 1$  for any 1-regular coherent configuration  $\mathcal{X}$ . In the following statement we will use the fact that the equality

$$(10) \quad b(\mathcal{X}) = b(\text{Aut}(\mathcal{X}))$$

holds for any cyclotomic scheme  $\mathcal{X}$  (statement (2) of [8, Theorem 1.2]).

**Theorem 3.2.** *Let  $\mathcal{X}$  be an antisymmetric cyclotomic scheme over a finite field. Then  $gb(\mathcal{X}) \leq 1$  and  $b(\mathcal{X}) \leq 3$ .*

**Proof.** We note that any antisymmetric scheme of degree  $> 1$  has rank at least 3. By the hypothesis this implies that  $\mathcal{X}$  is a proper cyclotomic scheme in the sense of [8]. Therefore by the McConnell theorem (inclusion (1) of this paper) this implies that  $\text{Aut}(\mathcal{X}) \leq \text{AGL}(1, \mathbb{F})$  where  $\mathbb{F}$  is the underlying finite field. Thus due to (10) we have

$$b(\mathcal{X}) = b(\text{Aut}(\mathcal{X})) \leq b(\text{AGL}(1, \mathbb{F})) \leq 3.$$

To prove that  $gb(\mathcal{X}) \leq 1$  set  $b = b(\mathcal{X})$ . Without loss of generality we can assume that  $b = 2$  or  $b = 3$ . Denote by  $\mathcal{Y}$  the  $\Pi$ -fission of  $\mathcal{X}$  with  $\Pi = \{B\}$  where  $B = \{\alpha_0, \dots, \alpha_{b-1}\}$  is a base of  $\mathcal{X}$ . Then it suffices to verify that

$$(11) \quad B \notin \Phi(\mathcal{Y}).$$

Indeed, in this case the set  $B$  must be the union of  $b$  fibers which are singletons because the size of any fiber of antisymmetric configuration is of odd cardinality. But then  $\mathcal{Y} = \text{Fis}(\mathcal{X}, B)$  is the complete configuration. Thus  $\Pi$  is a generalized base of  $\mathcal{X}$  and we are done.

To prove (11) suppose on the contrary that  $B \in \Phi(\mathcal{Y})$ . Then  $b = 3$  because any antisymmetric scheme, and hence  $\mathcal{Y}_B$ , has odd degree. However, up to isomorphism there is a unique antisymmetric scheme of degree 3, namely, the scheme of a regular group of order 3. This implies that  $r_{\mathcal{Y}}(\alpha_0, \alpha_1) = r_{\mathcal{Y}}(\alpha_0, \alpha_2)^*$ , and hence

$$(12) \quad r(\alpha_0, \alpha_1) = r(\alpha_0, \alpha_2)^*$$



On the other hand, by the transitivity of the group  $\text{Aut}(\mathcal{X})$  we can assume that  $\alpha_0 = 0_{\mathbb{F}}$ . Then it is easily seen that the set of fixed points of the two-point stabilizer  $\text{Aut}(\mathcal{X})_{\alpha_0, \alpha_1}$  is an additive subgroup of  $\mathbb{F}$ . So from (10) it follows that the set  $\{\alpha_0, \alpha_1, -\alpha_2\}$  is also a base of  $\mathcal{X}$ . Thus without loss of generality we can assume that

$$r(\alpha_0, \alpha_1) \neq r(\alpha_0, \alpha_2)^*.$$

However, this contradicts (12).  $\blacksquare$

**3.3. Bases of size at most 2.** A symmetric relation  $s \in S^{\cup}$  is called *connected* if any two distinct points in  $\Omega$  are joined by a path in the graph  $(\Omega, s)$ . It is well-known that a scheme  $\mathcal{X}$  is primitive if and only if any non-reflexive relation  $s \cup s^*$ ,  $s \in S$ , is connected.

**Theorem 3.3.** *Let  $\mathcal{X}$  be a coherent configuration and  $s \in S^{\cup}$  a connected relation. Suppose that for any point  $\alpha \in \Omega$  the coherent configuration  $(\mathcal{X}_{\alpha})_{\alpha s}$  is semiregular. Then any pair of distinct points in  $s$  forms a base of  $\mathcal{X}$ . In particular,  $b(\mathcal{X}) \leq 2$ .*

**Proof.** Without loss of generality we can assume that  $s \cap 1_{\Omega} = \emptyset$ . Let  $(\alpha, \beta) \in s$ . Set  $\Gamma = \{\gamma \in \Omega : \{\gamma\} \in \Phi(\mathcal{X}_{\alpha, \beta})\}$ . Then obviously  $\alpha, \beta \in \Gamma$ . Moreover, given  $\gamma \in \Gamma$  we have

$$(13) \quad \gamma s \subset \Gamma \quad \text{or} \quad \gamma s \cap \Gamma = \emptyset.$$

Indeed, suppose on the contrary that there exist points  $\gamma \in \Gamma$  and  $\gamma_1, \gamma_2 \in \gamma s$  such that  $\gamma_1 \in \Gamma$  and  $\gamma_2 \notin \Gamma$ . Since the coherent configuration  $(\mathcal{X}_{\gamma})_{\gamma s}$  is semiregular this implies that  $\{\gamma_2\} \in \Phi(\mathcal{X}_{\gamma, \gamma_1})$ . However, the coherent configuration  $\mathcal{X}_{\gamma, \gamma_1}$  is a fusion of  $\mathcal{X}_{\alpha, \beta}$  because  $\gamma, \gamma_1 \in \Gamma$ . Therefore  $\{\gamma_2\} \in \Phi(\mathcal{X}_{\alpha, \beta})$ , and hence  $\gamma_2 \in \Gamma$ . Contradiction.

Denote by  $\Gamma_0$  the set of all points  $\gamma \in \Gamma$  for which  $\gamma s \subset \Gamma$ . Then  $\alpha \in \Gamma_0$  because  $(\alpha, \beta) \in s$ ,  $\beta \in \Gamma$  and the coherent configuration  $(\mathcal{X}_{\alpha})_{\alpha s}$  is semiregular. By (13) this implies that  $\Gamma_0$  is the connectivity component of the graph  $(\Omega, s)$  that contains the vertex  $\alpha$ . Since this graph is connected, this implies that  $\Gamma_0 = \Omega$ . Therefore  $\Gamma = \Omega$ . By the definition of  $\Gamma$  this means any fiber of the coherent configuration  $\mathcal{X}_{\alpha, \beta}$  is singleton, and hence this configuration is complete. Thus  $\{\alpha, \beta\}$  is a base of  $\mathcal{X}$ .  $\blacksquare$

The following special statement will be used in the proof of Lemma 9.4. Below given a nonnegative integer  $m$  and relations  $r, s \in S$  we denote by  $r \circ_m s$  the set of all  $t \in r^* s$  such that  $c_{rt}^s \leq m$ .

**Lemma 3.4.** *Let  $\mathcal{X}$  be an antisymmetric primitive schurian scheme and  $r \in S$  a non-reflexive relation such that  $|r \circ_2 r \cup r \circ_2 r^*| > 2n_r/3$  and  $r \circ_2 r^* \neq \emptyset$ . Then  $b(\mathcal{X}) \leq 2$ .*

**Proof.** By the hypothesis  $s = r \cup r^*$  is a connected non-reflexive relation of  $\mathcal{X}$ . So by Theorem 3.3 it suffices to verify that the coherent configuration  $\mathcal{X}_0 = (\mathcal{X}_{\alpha})_{\alpha s}$  is semiregular for all  $\alpha \in \Omega$ . However, from the schurity of  $\mathcal{X}$  it follows that

$$\alpha r, \alpha r^* \in \Phi(\mathcal{X}_0).$$

Denote by  $S_1$  and  $S_2$  the set of thin relations in  $(S_0)_{\alpha r, \alpha r^*}$  and in  $(S_0)_{\alpha r}$  respectively. Then one can see that the coherent configuration  $\mathcal{X}_0$  is semiregular if and only if the following inequalities hold:

$$(14) \quad |S_1| > 0 \quad \text{and} \quad |S_2| > n_r/3.$$

Let  $u \in r \circ_2 r^*$ . Since  $n_u = n_{u^*}$ , from (4) we obtain that  $u^* \in r^* \circ_2 r$ . This implies that any point in  $\alpha r$  has at most two neighbors in the relation  $u' = u_{\alpha r^*, \alpha r}$ . On the other hand, the valences of  $\mathcal{X}_0$  are odd. Thus by Lemma 2.2 the relation  $u'$  contains a relation from  $S_1$ . This proves the first inequality in (14). Let now  $v \in r \circ_2 r$ . Then again any point in  $\alpha r$  has at most two neighbors in  $v' = v_{\alpha r, \alpha r}$ , and the above argument shows that  $v$  contains a relation from  $S_2$ . Thus by the lemma hypothesis we have

$$|S_1| + |S_2| \geq |r \circ_2 r \cup r \circ_2 r^*| > 2n_r/3.$$

Therefore either  $S_1$  or  $S_2$  contains more than  $n_r/3$  elements. In the latter case the second inequality in (14) is clear, whereas in the former case it follows because  $S_2$  contains a set  $t \cdot S_1^*$  where  $t$  is an arbitrary element from  $S_1$ . ■

**3.4. Bases and isomorphisms.** The following statement shows how bases are used to find isomorphisms between coherent configurations.

**Theorem 3.5.** *Let  $\mathcal{X}$  and  $\mathcal{X}'$  be coherent configuration on  $n$  points. Then given an algebraic isomorphism  $\varphi : \mathcal{X} \rightarrow \mathcal{X}'$  all the elements in the set  $\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$  can be listed in time  $(bn)^{O(b)}$  where  $b = b(\mathcal{X})$ .*

**Proof.** By exhaustive search in time  $n^{O(b)}$  one can find a size  $b$  base  $B$  of  $\mathcal{X}$ . Obviously, any isomorphism from  $\mathcal{X}$  onto  $\mathcal{X}'$  takes it to a base of  $\mathcal{X}'$ . Therefore

$$\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi) = \bigcup_{B'} \bigcup_g \text{Iso}_g(\mathcal{X}, \mathcal{X}', \varphi)$$

where  $B'$  runs over all size  $b$  point sets of  $\mathcal{X}'$ ,  $g$  runs over all bijections from  $B$  onto  $B'$ , and  $\text{Iso}_g(\mathcal{X}, \mathcal{X}', \varphi)$  consists of all  $f \in \text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$  such that  $f|_B = g$ . Since there are at most  $(bn)^{O(b)}$  possibilities for a pair  $(B', g)$ , only we need is to find in time  $n^{O(1)}$  the set  $\text{Iso}_g(\mathcal{X}, \mathcal{X}', \varphi)$  for fixed such a pair. To do this set

$$\mathcal{S} = \mathcal{S} \cup \{1_{\{\alpha\}} : \alpha \in B\} \quad \text{and} \quad \mathcal{S}' = \mathcal{S}' \cup \{1_{\{\alpha'\}} : \alpha' \in B'\}$$

where  $\mathcal{S}$  and  $\mathcal{S}'$  are the sets of basis relations of  $\mathcal{X}$  and  $\mathcal{X}'$  respectively. Then obviously the coherent configurations

$$\text{Fis}(\mathcal{S}) = \text{Fis}(\mathcal{X}, B) \quad \text{and} \quad \text{Fis}(\mathcal{S}') = \text{Fis}(\mathcal{X}', B')$$

are complete. So any algebraic isomorphism between them is induced by exactly one bijection between their fiber sets. Thus the required statement immediate follows from Theorem 2.4 for the bijection  $\psi : \mathcal{S} \rightarrow \mathcal{S}'$  defined by the following conditions:  $\psi|_{\mathcal{S}} = \varphi$  and  $\psi(1_{\{\alpha\}}) = 1_{\{\alpha'g\}}$  for all  $\alpha \in B$ . ■

The following technical notion was introduced in [9, Section 3.2]. Let  $\mathcal{X}$  be a scheme and  $e_0, e_1 \in S^\cup$  two equivalence relations such that  $e_0 \subset e_1$ . Set

$$\Omega_0 = \{\Gamma_1/e_0 : \Gamma_1 \in \Omega/e_1\}.$$

By a *majorant* of the group  $G = \text{Aut}(\mathcal{X})$  with respect to the pair  $(e_0, e_1)$  we mean a permutation group  $H$  on a set  $\Delta$  together with a family of bijections  $f_\Gamma : \Gamma \rightarrow \Delta$  where  $\Gamma \in \Omega_0$ , such that

$$(15) \quad (G^\Gamma)^{f_\Gamma} \leq H$$

where  $G^\Gamma = G^{\Gamma_1/e_0}$  is the permutation group induced by the natural action of the setwise stabilizer  $G_{\{\Gamma_1\}}$  on the set  $\Gamma$ .

**Corollary 3.6.** *In the above notation let  $\Gamma$  be an element of  $\Omega_0$ . Suppose that*

$$(16) \quad \text{Iso}(\mathcal{X}_\Gamma, \mathcal{X}_{\Gamma'}, \varphi_{\Gamma, \Gamma'}) \neq \emptyset \quad \text{for all } \Gamma' \in \Omega_0$$

where  $\varphi_{\Gamma, \Gamma'}$  is the algebraic isomorphism (6) for  $\mathcal{X} = \mathcal{X}_{\Omega/e_0}$ . Then the group  $\text{Aut}(\mathcal{X}_\Gamma)$  together with any family of bijections  $f_{\Gamma'} \in \text{Iso}(\mathcal{X}_\Gamma, \mathcal{X}_{\Gamma'}, \varphi_{\Gamma, \Gamma'})$ ,  $\Gamma' \in \Omega_0$ , is a majorant of  $\text{Aut}(\mathcal{X})$  with respect to the pair  $(e_0, e_1)$ . Moreover, it can be constructed in time  $(bn)^{O(b)}$  where  $n = |\Gamma|$  and  $b = b(\mathcal{X}_\Gamma)$ .

**Proof.** For any  $\Gamma' \in \Omega_0$  we obviously have inclusion  $G^{\Gamma'} \leq \text{Aut}(\mathcal{X}_{\Gamma'})$ . On the other hand, given a bijection  $f_{\Gamma'} \in \text{Iso}(\mathcal{X}_\Gamma, \mathcal{X}_{\Gamma'}, \varphi_{\Gamma, \Gamma'})$ , we have

$$\text{Aut}(\mathcal{X}_{\Gamma'})^{f_{\Gamma'}} = \text{Aut}(\mathcal{X}_\Gamma^{f_{\Gamma'}}) = \text{Aut}(\mathcal{X}_\Gamma).$$

Thus inclusion (15) holds for  $\Delta = \Gamma$  and  $H = \text{Aut}(\mathcal{X}_\Gamma)$ . This proves the first statement of the lemma. The second one follows from Theorem 3.5.  $\blacksquare$

#### 4. PRIMITIVE LINEAR GROUPS OF ODD ORDER

The structure of a finite solvable linear primitive group was studied in [21, 17]. The following theorem is just a specialization of [22, Theorem 2.2] for the groups of odd order.

**Theorem 4.1.** *Let  $K \leq \text{GL}(d, p)$  be a primitive group of odd order. Then every normal abelian subgroup of  $K$  is cyclic and  $K$  has a series  $1 < U \leq F \leq A \leq K$  of normal subgroups such that the following statements hold:*

- (P1)  $\text{Span}(U) = \text{GF}(p^a)$  where  $a$  is a divisor of  $d$ ,
- (P2)  $C_K(F) \leq F \leq \text{Fit}(K)$  and  $|F : U| = e^2$  for some integer  $e$  such that each prime divisor of  $e$  divides  $p^a - 1$ ,
- (P3)  $A = C_K(U)$  and  $A/F$  is isomorphic to a completely reducible subgroup of the group  $\prod_{i=1}^m \text{Sp}(2n_i, p_i)$  where  $p_i$  and  $n_i$  are defined from the prime power decomposition  $e = \prod_{i=1}^m p_i^{n_i}$ ,
- (P4)  $|K : A|$  divides  $a$  and  $ae$  divides  $d$ .  $\blacksquare$

From statements (P1) and (P4) it follows that  $|U| \leq u_{a,p}$  and  $|K : A| \leq a_0$  where  $u_{a,p}$  and  $a_0$  are the maximal odd divisors of  $p^a - 1$  and  $a$  respectively. Thus

$$(17) \quad |K| \leq u_{a,p} \cdot e^2 \cdot s_e \cdot a_0$$

where  $s_e$  is the maximal order of the group  $A/F$  for a fixed  $e$  (see statement (P3)). The following two lemmas collect some special facts on the group  $K$  from Theorem 4.1 that are contained in papers [12, 13] or obtained by means of computer package GAP [16].

**Lemma 4.2.** *Let  $e$  be the number from Theorem 4.1. Then one of the following statements hold:*

- (1)  $e = 1$  and  $K \leq \Gamma\text{L}(1, p^d)$ ,
- (2)  $e \in \{5, 9, 11, 13\}$  and  $s_5 \leq |K : F| = 3$ ,  $s_9, s_{11} \leq 5$ ,  $s_{13} \leq 7$ ,
- (3)  $e \geq 15$  is an odd integer and  $s_e \leq e^2/2$ .

**Proof.** Suppose first that  $e = 1$ . Then from (P2) it follows that  $U = F$  is a normal abelian self-centralizing subgroup of  $K$ . By [17, Lemma 2.2] this implies that  $F$  is irreducible. Thus by [17, Theorem 2.1] we conclude that  $K \leq \Gamma\text{L}(1, p^d)$  which proves the second part of statement (1). For  $e \geq 15$  the required inequality

immediately follows from the fact that any completely reducible odd order subgroup of the group  $\mathrm{Sp}(2n_i, p_i)$  has a regular orbit on the underground linear space (see [12, Theorem A]). To deal with the case  $1 < e < 15$  we start with some observation.

Suppose that  $e$  is an odd prime. We claim that the group  $K/F$  has an irreducible representation in  $\mathrm{GL}(2, e)$ . Indeed, by statement (1) of [22, Theorem 2.2] the group  $F$  is a central product of  $U$  and a characteristic subgroup  $E$  of  $K$  that contains an extraspecial subgroup  $E_0$  of order  $e^3$  and exponent  $e$ . In particular,  $Z = E \cap U$  is a central subgroup of  $F$ ,  $E/Z \cong F/U$  and  $|E_0 \cap Z| \leq e$ . Therefore by (P2) we have  $|E : Z| = |F : U| = e^2$ , and

$$(18) \quad F/U \cong E_0/(E_0 \cap Z) \cong \mathbb{Z}_e \times \mathbb{Z}_e.$$

However, by statement (2) of [22, Theorem 2.2] the group  $F/U$  is a completely reducible  $K/F$ -module. Therefore  $K/F$  has a representation in  $\mathrm{GL}(2, e)$ . Suppose that this representation is not irreducible. Then one can find a group  $E' > Z$  such that  $|E : E'| = e$  and the group  $E'/Z$  is  $K/F$ -invariant. But then obviously  $E'$  is a normal abelian subgroup of  $K$ . Moreover from (18) it follows that  $|E' \cap E_0| = e^2$ . Therefore  $E'$  contains an elementary abelian subgroup of order  $e^2$ . Thus  $E'$  is normal abelian non-cyclic subgroup of  $K$ , which is impossible by Theorem 4.1. The claim is proved.

Let now  $1 < e < 15$ . By means of GAP we find that (a) there are no odd order irreducible subgroups in  $\mathrm{GL}(2, e)$  for  $e = 3, 7$ , (b) the maximal order of an irreducible odd order subgroup in  $\mathrm{GL}(2, e)$  for  $e = 5, 11, 13$  equals respectively to 3, 15 and 21, and (c) the irreducible subgroups in  $\mathrm{GL}(2, 11)$  of order 15 and in  $\mathrm{GL}(2, 13)$  of order 21 are not subgroups of  $\mathrm{Sp}(2, 11)$  and  $\mathrm{Sp}(2, 13)$  respectively. Thus the required statement immediately follows from the above claim unless  $e = 9$ . In the remaining case the same argument as in the claim shows that  $K/F$  has a representation in  $\mathrm{GL}(4, e)$ . Since up to conjugacy the latter group has a unique irreducible odd order subgroup and the order of it is 5, it suffices to verify that that representation is irreducible. Suppose that this is not true. Then as in the above claim one can check that there is no  $K/F$ -invariant subgroup of  $E/Z$  of order  $e$ . Therefore such a subgroup has order  $e^2$ . But this is impossible by statement (a) with  $e = 3$ .  $\blacksquare$

**Lemma 4.3.** *In the notation of Theorem 4.1 we have  $\mathrm{fix}(K) \leq p^{\lfloor 4d/9 \rfloor}$ . Moreover, if  $g$  is an element of  $K$  of prime order  $q$ , then*

- (1)  $\mathrm{fix}(g) \leq p^{\lfloor d/q \rfloor}$  for  $g \in F$ ,
- (2)  $\mathrm{fix}(g) \leq p^{\lfloor d/3 \rfloor}$  for  $g \notin F$  and  $q \neq 3$ .

**Proof.** Follows from Lemma 1.3 of [12] and the proof of it.  $\blacksquare$

## 5. BASES OF THE WREATH PRODUCT

In this section we fix a coherent configuration  $\mathcal{X}_i = (\Omega_i, S_i)$ ,  $i = 1, 2$ . The *wreath product*  $\mathcal{X}_1 \wr \mathcal{X}_2$  can be defined as the smallest coherent configuration  $\mathcal{X} = (\Omega, S)$  with  $\Omega = \Omega_1 \times \Omega_2$  such that the set  $S^\cup$  contains the equivalence relation  $e$  with classes  $\Omega_\alpha = \Omega_1 \times \{\alpha\}$ ,  $\alpha \in \Omega_2$ , and

$$(\mathcal{X}_{\Omega_\alpha})^{\pi_\alpha} = \mathcal{X}_1, \quad \mathcal{X}^\pi = \mathcal{X}_2$$

for all  $\alpha$  where  $\pi_\alpha : \Omega_\alpha \rightarrow \Omega_1$  and  $\pi : \Omega \rightarrow \Omega_2$  are the natural projections. In particular,  $\pi_\alpha \in \text{Iso}(\mathcal{X}_{\Omega_\alpha}, \mathcal{X}_1)$  and  $\mathcal{X}_{\Omega/e} = \mathcal{X}_2$ . When the coherent configuration  $\mathcal{X}$  is homogeneous, we have

$$(19) \quad S = \{s_1 \otimes 1_{\Omega_2} : s_1 \in S_1\} \cup \{\Omega_1^2 \otimes s_2 : s_2 \in S_2, s_2 \neq 1_{\Omega_2}\}.$$

Any imprimitive schurian scheme is isomorphic to a fission of the wreath product of two smaller schemes. In general case the set  $\Phi(\mathcal{X})$  consists of all sets  $\Gamma_1 \times \Gamma_2$  where  $\Gamma_1 \in \Phi(\mathcal{X}_1)$  and  $\Gamma_2 \in \Phi(\mathcal{X}_2)$ , and

$$(20) \quad \mathcal{X}_{\Gamma_1 \times \Gamma_2} = (\mathcal{X}_1)_{\Gamma_1} \wr (\mathcal{X}_2)_{\Gamma_2}.$$

**Lemma 5.1.** *Let  $\mathcal{X} = \mathcal{X}_1 \wr \mathcal{X}_2$  and  $\Pi \subset 2^\Omega$ . Suppose that*

- (1)  $\Pi_\alpha = \{\Gamma \cap \Omega_\alpha : \Gamma \in \Pi\}$  *is a generalized base of  $\mathcal{X}_{\Omega_\alpha}$  for all  $\alpha \in \Omega_2$ ,*
- (2)  $\Pi_{\Omega/e} = \{\Gamma^\pi : \Gamma \in \Pi\}$  *is a generalized base of  $\mathcal{X}_2$ .*

*Then  $\Pi$  is a generalized base of  $\mathcal{X}$ .*

**Proof.** Set  $\mathcal{Y} = \text{Fis}(\mathcal{X}, \Pi)$ . Then obviously  $\mathcal{Y}^\pi \geq \text{Fis}(\mathcal{X}_2, \Pi_2)$  is a complete configuration by condition (2). This implies that  $\Omega_\alpha \in \Phi(\mathcal{Y})^\cup$  for all  $\alpha \in \Omega_2$ . It follows that  $\Gamma \cap \Omega_\alpha \in \Phi(\mathcal{Y})^\cup$  for all  $\Gamma \in \Pi$ . Therefore  $\mathcal{Y}_{\Omega_\alpha} \geq \text{Fis}(\mathcal{X}_{\Omega_\alpha}, \Pi_\alpha)$  is a complete configuration for all  $\alpha$ . Consequently, any fiber of  $\mathcal{Y}$  is a singleton, which means that the coherent configuration  $\mathcal{Y}$  is complete. Thus  $\Pi$  is a generalized base of  $\mathcal{X}$ . ■

Let  $\Pi$  be a generalized base of the coherent configuration  $\mathcal{X}$ . We say that  $\Pi$  is *proper* if there exists a set  $\Gamma \in \Pi$  such that  $\Gamma \cap \Omega_\alpha$  is a proper subset of  $\Omega_\alpha$  for all  $\alpha \in \Omega_2$ . Clearly, such a base can exist only if  $|\Omega_2| > 1$ .

**Theorem 5.2.** *Let  $\mathcal{X} = \mathcal{X}_1 \wr \mathcal{X}_2$  and  $b = \max\{gb(\mathcal{X}_1), gb(\mathcal{X}_2)\}$ . Suppose that  $\mathcal{X}_1$  is antisymmetric. Then  $gb(\mathcal{X}) \leq b$ . Moreover, if  $b > 0$ , then there exists a proper generalized base of  $\mathcal{X}$  of size  $b$ .*

**Proof.** Without loss of generality we can also assume that  $|\Omega_1| > 1$  and  $b > 0$ , and that the coherent configurations  $\mathcal{X}_1$ ,  $\mathcal{X}_2$ , and hence  $\mathcal{X}$ , are homogeneous (see the first inequality in (9) and equality (20)). Let  $\Pi_i$  be a generalized base of  $\mathcal{X}_i$  of size  $b$ ,  $i = 1, 2$ . The assumption implies that the set  $\Pi_1$  contains a proper subset of  $\Omega_1$ . Let us choose a bijection  $\Gamma_1 \mapsto \Gamma_2$  from  $\Pi_1$  onto  $\Pi_2$ , and denote by  $\Pi$  the set of all

$$(21) \quad \Gamma = \Gamma_1 \times \Gamma_2 \cup \Gamma'_1 \times \Gamma'_2$$

with  $\Gamma_1 \in \Pi_1$  where  $\Gamma'_i$  is the complement to  $\Gamma_i$  in  $\Omega_i$ ,  $i = 1, 2$ . Then  $|\Pi| = b$ . So it suffices to verify that  $\Pi$  is a generalized base of  $\mathcal{X}$  (in this case  $\Pi$  is proper because  $\Pi_1$  contains a proper subset of  $\Omega_1$ ).

One can see that conditions (1) and (2) of Lemma 5.1 are satisfied for the union of  $\Pi$  and  $\Pi' = \{\Gamma_1 \times \Gamma_2 : \Gamma_1 \in \Pi_1\}$ . So by this lemma the union is a generalized base of  $\mathcal{X}$ . Thus we have to verify only that  $\text{Fis}(\mathcal{X}, \Pi \cup \Pi')$  is a fission of  $\mathcal{Y} = \text{Fis}(\mathcal{X}, \Pi)$ , or, equivalently, that

$$(22) \quad \Gamma_1 \times \Gamma_2 \in \Phi(\mathcal{Y})^\cup$$

for all  $\Gamma_1 \in \Pi_1$ . To do this denote by  $e'$  the equivalence relation on the the set  $\Gamma$  such that  $\Gamma/e' = I \cup I'$  with

$$I = \{\Gamma \cap \Omega_\alpha : \alpha \in \Gamma_2\} \quad \text{and} \quad I' = \{\Gamma \cap \Omega_\alpha : \alpha \in \Gamma'_2\}.$$

Then from (21) it follows that  $e' = \Gamma^2 \cap e$  is also a relation of  $\mathcal{Y}$ . Besides, since  $\mathcal{X}_1$  is antisymmetric, exactly one of the numbers  $\Gamma_1$  and  $\Gamma'_1$  is odd. This implies that the hypothesis of Lemma 2.1 is satisfied for  $\mathcal{X} = \mathcal{Y}$  and  $e = e'$ . By this lemma the union of all elements of  $I$  belongs to the set  $\Phi(\mathcal{Y})^\cup$ . Since the union is obviously equal to  $\Gamma_1 \times \Gamma_2$ , we conclude that (22) holds.  $\blacksquare$

Let  $\Pi$  be a proper generalized base of the coherent configuration  $\mathcal{X}$ . We say that  $\Pi$  is *thin* if  $|\Gamma \cap \Omega_\alpha| \leq 1$  for all  $\Gamma \in \Pi$  and  $\alpha \in \Omega_2$ . The following statements will be used in Section 7 to estimate the base size of an exponentiation.

**Theorem 5.3.** *Let  $\mathcal{X} = \mathcal{X}_1 \wr \mathcal{X}_2$ . Suppose that  $\mathcal{X}_1$  is antisymmetric. Then  $\mathcal{X}$  has a thin generalized base of size  $b = b_1 + \max\{0, b_2 - \lceil b_1/2 \rceil\}$  where  $b_1 = b(\mathcal{X}_1)$  and  $b_2 = gb(\mathcal{X}_2)$ .*

**Proof.** Let  $\Pi_1$  be a base of  $\mathcal{X}_1$  of size  $b_1$ , and  $\Pi_2$  a generalized base of  $\mathcal{X}_2$  of size  $b_2$ . Suppose first that  $2b_2 \geq b_1$ . Then  $b = \lceil b_1/2 \rceil + b_2$ . Without loss of generality we can assume that  $b_1$  is even (otherwise we add an extra point to  $\Pi_1$ ). Let us fix

- a point  $\delta \in \Omega_1$ ,
- a decomposition  $\Pi_1 = B \cup B'$  into two disjoint sets of equal size,
- a fixed point free involution  $\beta \mapsto \beta'$  on  $\Omega_1$  taking  $B$  to  $B'$ ,
- an injection  $B \rightarrow \Pi_2$ ,  $\beta \mapsto \Gamma_\beta$ ; set  $\Pi'_2$  to be the complement to its image.

Denote by  $\Pi$  the family of sets  $\Gamma$  and  $\Gamma'$  defined below for all  $\beta \in B$ , and sets  $\{\delta\} \times \Gamma_2$  for all  $\Gamma_2 \in \Pi'_2$ ,

$$(23) \quad \Gamma = \{\beta\} \times \Gamma_\beta \cup \{\beta'\} \times \Gamma'_\beta \quad \text{and} \quad \Gamma' = \{\beta'\} \times \Gamma_\beta \cup \{\beta\} \times \Gamma'_\beta.$$

Since  $|B| = b_1/2$  and  $|\Pi'_2| = b_2 - b_1/2$ , the family  $\Pi$  is of size  $b = b_1/2 + b_2$ . To complete the proof we will verify that  $\Pi$  is a generalized base of  $\mathcal{X}$  (in this case  $\Pi$  is thin just by the definition).

To prove that the coherent configuration  $\mathcal{Y} = \text{Fis}(\mathcal{X}, \Pi)$  is complete, we note that  $\Phi(\mathcal{Y})^\cup$  contains the sets  $\Gamma^* = \Gamma \cup \Gamma'$  where  $\Gamma$  and  $\Gamma'$  are defined by (23). We claim that

$$(24) \quad \{\beta\} \times \Gamma_\beta \in \Phi(\mathcal{Y})^\cup$$

for all  $\beta \in B$ . Then obviously  $\{\beta'\} \times \Gamma'_\beta \in \Phi(\mathcal{Y})^\cup$ . This implies that conditions (1) and (2) of Lemma 5.1 are satisfied for  $\mathcal{X}$  and  $\Pi^*$  where the latter consists of all sets  $\{\beta\} \times \Gamma_\beta$ ,  $\{\beta'\} \times \Gamma'_\beta$  and  $\{\delta\} \times \Gamma_2$ . So by this lemma  $\Pi^*$  is a generalized base of  $\mathcal{X}$ . Thus the coherent configuration  $\mathcal{Y} \geq \text{Fis}(\mathcal{X}, \Pi^*)$  is complete and we are done.

To prove (24) suppose on the contrary that there is a set  $\Delta \in \Phi(\mathcal{Y})$  such that

$$(25) \quad \beta\alpha, \beta'\alpha' \in \Delta$$

for some  $\beta \in B$ ,  $\alpha \in \Gamma_\beta$  and  $\alpha' \in \Gamma'_\beta$ , where  $\beta\alpha = (\beta, \alpha)$  and  $\beta'\alpha' = (\beta', \alpha')$ . Denote by  $e^*$  the equivalence relation on  $\Gamma^*$  with classes  $\Gamma^* \cap \Omega_\gamma = \{\beta, \beta'\} \times \{\gamma\}$  where  $\gamma \in \Omega_2$ . Then  $e^* = e \cap (\Gamma^*)^2$  is a relation of  $\mathcal{Y}$ . Therefore the set  $\Delta' = \Delta e^*$  belongs to  $\Phi(\mathcal{Y})^\cup$ . Since the relation  $u := e^* \setminus 1_{\Gamma^*}$  is thin, this implies that  $\Delta'$  is a fiber of  $Y$  and  $u_{\Delta, \Delta'}$  is a basic relation of  $\mathcal{Y}$ . Thus from (25) we obtain that

$$r_{\mathcal{Y}}(\beta\alpha, \beta'\alpha') = r_{\mathcal{Y}}(\beta'\alpha', \beta\alpha) = u_{\Delta, \Delta'}.$$

However, in this case  $r_{\mathcal{X}_1}(\beta, \beta') = r_{\mathcal{X}_1}(\beta', \beta)$  which is impossible because the coherent configuration  $\mathcal{X}_1$  is antisymmetric.

Let now  $2b_2 < b_1$ . Then  $b = b_1$ . In this case take two disjoint sets  $B, B' \subset \Pi_1$  of the same size  $b_2$ , choose a bijection from  $B \rightarrow \Pi_2$ ,  $\beta \mapsto \Gamma_\beta$ , and set  $\Pi'_2$  to be the family of  $b_1 - 2b_2$  sets  $\{\beta\} \times \Omega_2$  where  $\beta$  runs over the set  $\Pi_1 \setminus (B \cup B')$ . Then the rest of the proof is completely analogous to the previous case.  $\blacksquare$

## 6. EXPONENTIATION

Let  $\Gamma$  be a finite set,  $m$  a positive integer and  $\Delta = \{1, \dots, m\}$ . Given a set  $T \subset 2^{\Gamma \times \Gamma}$  denote by  $T^{\otimes m}$  the set of all relations  $t_1 \otimes \dots \otimes t_m$  with  $t_i \in T$  for all  $i$ . For a coherent configuration  $\mathcal{Y} = (\Gamma, T)$  the pair

$$\mathcal{Y}^{\otimes m} = (\Gamma^m, T^{\otimes m})$$

is also a coherent configuration (the Cartesian  $m$ -power of  $\mathcal{Y}$ ). Any permutation group  $L \leq \text{Sym}(\Delta)$  has the natural action on  $\Omega = \Gamma^m$ : a permutation  $l \in L$  moves a point  $\alpha = (\dots, \alpha_i, \dots)$  to the point  $\alpha^l = (\dots, \alpha_j, \dots)$  with  $j^l = i$  (and hence a relation  $t = \dots \otimes t_i \otimes \dots$  to the relation  $t^l = \dots \otimes t_j \otimes \dots$ ). Denote by  $T \uparrow L$  the set of all relations  $t^L = \cup_{l \in L} t^l$  with  $t \in T^{\otimes m}$ . Then the pair

$$(26) \quad \mathcal{X} = \mathcal{Y} \uparrow L = (\Omega, T \uparrow L)$$

is a coherent configuration [5] called the *exponentiation* of  $\mathcal{Y}$  by  $L$ .<sup>6</sup> It was also proved in that paper that  $\mathcal{X}$  is schurian if and only if so is  $\mathcal{Y}$ , and that  $\mathcal{X}$  is primitive if and only if  $L$  is transitive and  $\mathcal{Y}$  is primitive and non-regular. It is easily seen that  $\mathcal{X} = \mathcal{Y}$  whenever  $m = 1$ .

In this paper we will use the exponentiation construction for the scheme of a primitive solvable permutation group. The structure of such a group is described in Theorem 2.3. Depending on whether the group  $K$  from this theorem is primitive (as a linear group) or not we will say that the scheme  $\mathcal{X} = \text{Inv}(G)$  is *linearly primitive* or *linearly imprimitive*. In particular, in both cases  $\mathcal{X}$  is schurian and the following statement holds.

**Theorem 6.1.** *The scheme  $\mathcal{X}$  has a (possibly trivial) fusion isomorphic to  $\mathcal{Y} \uparrow L$  where  $\mathcal{Y}$  is a linearly primitive scheme and  $L$  is a transitive group. Moreover, if  $\mathcal{X}$  is antisymmetric, then  $\mathcal{Y}$  is antisymmetric and  $L$  has odd order.*

**Proof.** Without loss of generality we can assume that the group  $K$  is imprimitive. Then the linear space  $\Omega$  is a direct sum of the subspaces belonging to the set

$$\Delta = \{\Gamma^k : k \in K\}$$

where  $\Gamma$  is a proper subspace of  $\Omega$ , and  $K$  is isomorphic to a subgroup of the wreath product of the group  $K^U = (K_{\{U\}})^U \leq \text{GL}(U)$  and the transitive permutation group  $K^\Delta \leq \text{Sym}(\Delta)$  induced by the action of  $K$  on  $\Delta$ , [21, Section 15.2]. According to [7, Proposition 4.1] this implies that  $G$  can be identified with a subgroup of the wreath product  $G^U \uparrow K^\Delta$  of permutation groups  $G^U$  and  $K^\Delta$  in primitive action. On the other hand, by [15, p.212] we have

$$\text{Inv}(G^U \uparrow K^\Delta) = \text{Inv}(G^U) \uparrow K^\Delta$$

Thus the scheme  $\mathcal{X} = \text{Inv}(G)$  has a fusion  $\mathcal{Y} \uparrow L$  where  $\mathcal{Y} = \text{Inv}(G^U)$  and  $L = K^\Delta$ . Moreover, if the scheme  $\mathcal{Y}$  is linearly imprimitive, then by the above it has a fusion  $\mathcal{Y}' \uparrow L'$  for some scheme  $\mathcal{Y}' = \text{Inv}(G')$  where  $G'$  is a primitive solvable permutation

<sup>6</sup>It is a special case of the general construction of the exponentiation introduced in [5].

group and  $L'$  is a transitive group. So by [7, Proposition 3.3] the scheme  $\mathcal{X}$  has a fusion  $(\mathcal{Y}' \uparrow L') \uparrow L = \mathcal{Y}' \uparrow (L' \wr L)$  and the first statement follows. To prove the second statement it suffices to note that if the scheme  $\mathcal{X}$  is antisymmetric, then the group  $K$  has odd order.  $\blacksquare$

## 7. BASES OF THE EXPONENTIATION.

The following theorem gives upper bounds for the maximal sizes of generalized and ordinary bases of the exponentiation (26) when the coherent configuration  $\mathcal{Y}$  is antisymmetric. The former bound is the best possible whereas the latter one definitely not. Nevertheless, even this rather weak bound is sufficient for the purpose of the paper.

**Theorem 7.1.** *Let  $\mathcal{Y}$  be an antisymmetric coherent configuration and let  $L$  be a transitive permutation group of odd order. Then  $gb(\mathcal{Y} \uparrow L) \leq \max\{gb(\mathcal{Y}), b\}$  where  $b = gb(\text{Inv}(L))$ . Moreover, if  $\mathcal{Y}$  is not complete, then*

$$b(\mathcal{Y} \uparrow L) \leq b(\mathcal{Y}) + \max\{0, b - \lceil (b(\mathcal{Y}) - 1)/2 \rceil\}.$$

The proof of Theorem 7.1 will be given in the end of this section. Let us fix some notations. Let  $\mathcal{X} = (\Omega, S)$  be the coherent configuration defined by (26). For any  $i \in \{0, \dots, m\}$  set

$$r_i = \{(\alpha, \beta) \in \Gamma^m \times \Gamma^m : d(\alpha, \beta) = i\}$$

where  $d(\alpha, \beta)$  is the number of all  $j \in \Delta$  such that  $\alpha_j \neq \beta_j$ . One can see that  $r_i$  is the union of the relations from  $T^{\otimes m}$  in which  $i$  factors are equal to  $1_\Gamma$  and the other  $m - i$  are  $\Gamma^2 \setminus 1_\Gamma$ . Therefore  $r_i \in S^\cup$  for all  $i$  (which means that  $\mathcal{X}$  is a fission of a Hamming scheme). In what follows we set  $r_{-1} = \emptyset$  and  $r = r_1$ .

Let us fix a point  $\gamma_0 \in \Gamma$ , and set  $\alpha = \alpha(\gamma_0)$  to be the point of  $\Omega$  with all coordinates equal to  $\gamma_0$ . Then the neighborhood  $\alpha r$  of  $\alpha$  in  $r$  is the disjoint union of the sets

$$(27) \quad \Gamma_i = \{\beta \in \Omega : d(\alpha, \beta) = 1 \text{ and } \beta_i \neq \gamma_0\}, \quad i \in \Delta.$$

They are the classes of an equivalence relation on  $\alpha r$  that is denoted by  $e$ . It is easily seen that  $e = 1_{\alpha r} \cup r_{\alpha r}$ . Therefore  $e$  is a relation of the coherent configuration  $\mathcal{X}_0 = (\mathcal{X}_\alpha)_{\alpha r}$ . The following two lemmas are key ingredients in our proof.

**Lemma 7.2.** *The mapping  $\rho : \Omega \rightarrow 2^{\alpha r}$ ,  $\beta \mapsto \beta r_{d-1} \cap \alpha r$  where  $d = d(\alpha, \beta)$ , is an injection and*

$$\text{Im}(\rho) = \{\Lambda \subset \alpha r : |\Lambda \cap \Gamma_i| \leq 1 \text{ for all } i \in \Delta\}.$$

*In particular, the set  $\alpha r$  is a base of the coherent configuration  $\mathcal{X}_\alpha$ .*

**Proof.** Given  $\beta \in \Omega$  and  $i \in \Delta$  such that  $\beta_i \neq \gamma_0$  set  $\beta^{(i)}$  to be the unique point in  $\Gamma_i$  the  $i$ th coordinate of which is equal to  $\beta_i$ . Then obviously

$$d(\beta, \beta^{(i)}) = d(\beta, \alpha) - 1.$$

Therefore  $\beta^{(i)} \in \rho(\beta)$ . On the other hand, let  $\delta \in \rho(\beta)$ . Then  $d(\delta, \beta) = d - 1$ . So the points  $\delta$  and  $\beta$  have exactly  $m - d + 1$  equal coordinates. At least  $m - d$  of them equal  $\gamma_0$ . But  $\beta$  has exactly  $m - d$  such coordinates. Therefore there is  $i \in \Delta$  such that  $\beta_i \neq \gamma_0$  and  $\beta_i = \delta_i$ . This means that  $\delta = \beta^{(i)}$ . Thus

$$(28) \quad \rho(\beta) = \{\beta^{(i)} : i \in \Delta, \beta_i \neq \gamma_0\}$$



which proves the first statement. To prove the second one it suffices to note that no two points  $\beta$  and  $\beta'$  with  $\rho(\beta) \neq \rho(\beta')$  belong the same fiber of the coherent configuration  $\text{Fis}(\mathcal{X}_\alpha, \alpha r)$ .  $\blacksquare$

Set  $\Gamma_0 = \Gamma \setminus \{\gamma_0\}$ . Let us define the mapping  $f : \Gamma_0 \times \Delta \rightarrow \alpha r$  taking a pair  $(\gamma, i)$  to the unique point  $\beta \in \Gamma_i$  for which  $\beta_i = \gamma$ . Then obviously  $f$  is a bijection and the  $f$ -image of the set  $\Gamma_0 \times \{i\}$  coincides with  $\Gamma_i$  for all  $i \in \Delta$ .

**Lemma 7.3.** *Set  $\mathcal{Y}_0$  to be the restriction of  $\mathcal{Y}_{\gamma_0}$  to  $\Gamma_0$ . Then  $\mathcal{X}_0^{f^{-1}} \geq \mathcal{Y}_0 \wr \text{Inv}(L)$ .*

**Proof.** Denote by  $T_0$  the set of all relations  $t_{\Gamma_0}$  with  $t \in T$  (we recall that  $T$  is the set of basis relations of  $\mathcal{Y}$ ). Then it is easily seen that  $\mathcal{Y}_0 = \text{Fis}(T_0)$ . So by the definition of wreath product it suffices to verify that for all  $t_0 \in T_0$  and all orbits  $u \in \text{Orb}(L, \Delta^2)$  we have

$$(29) \quad (t_0 \otimes 1_\Delta)^f, (\Gamma_0^2 \otimes u)^f \in S_0^\cup$$

where  $S_0$  is the set of basis relations of  $\mathcal{X}_0$ . To do this let  $t_0 \in T_0$ . Then  $t_0 = t_{\Gamma_0}$  for some  $t \in T$ . By the definition of the exponentiation and the transitivity of  $L$  the set  $S$  contains the relation

$$(30) \quad (t \otimes 1_\Delta \otimes \cdots \otimes 1_\Delta)^L = (t \otimes 1_\Delta \otimes \cdots \otimes 1_\Delta) \cup \cdots \cup (1_\Delta \otimes \cdots \otimes 1_\Delta \otimes t).$$

Denote by  $s$  the restriction of this relation to  $\alpha r$ . Then  $s \in S_0^\cup$  by Lemma 2.2. On the other hand, given  $i \in \Delta$  denote by  $s_i$  the summand in the right-hand side of (30) with  $t$  being at the  $i$ th position. Then a straightforward computation shows that  $(s_i)_{\alpha r}$  coincides with the  $f$ -image of  $t_0 \otimes 1_{\{i\}}$ . It follows that the relation

$$(t_0 \otimes 1_\Delta)^f = \bigcup_{i \in \Delta} (t_0 \otimes 1_{\{i\}})^f = \bigcup_{i \in \Delta} s_i = s$$

belongs to  $S_0^\cup$  which proves the first part of (29). To prove the second part let  $u \in \text{Orb}(L, \Delta^2)$ . Then  $S^\cup$  contains the union of relations  $u_{ij} = s_1 \otimes \cdots \otimes s_m$  with  $s_i = u$ ,  $s_j = u^*$  and  $s_k = 1_\Gamma$  for all  $k \neq i, j$ . Denote by  $s$  the restriction of this relation to  $\alpha r$ . Then  $s \in S_0^\cup$  by Lemma 2.2. On the other hand, a straightforward computation shows that given  $(i, j) \in u$  the set  $s_{ij} = (u_{ij})_{\alpha r}$  coincides with the  $f$ -image of the relation  $(\gamma_0 u \times \{i\}) \times (\gamma_0 u^* \times \{j\})$ . It follows that  $e \cdot s_{ij} \cdot e = \Gamma_i^2 \cup \Gamma_j^2 \cup \Gamma_i \times \Gamma_j$ . Thus the relation

$$(\Gamma_0^2 \otimes u)^f = \bigcup_{(i,j) \in u} ((\Gamma_0 \times \{i\}) \times (\Gamma_0 \times \{j\}))^f = \bigcup_{(i,j) \in u} (e \cdot s_{ij} \cdot e) \setminus e = (e \cdot s \cdot e) \setminus e$$

belongs to  $S_0^\cup$ , and we are done.  $\blacksquare$

**Proof of Theorem 7.1.** To prove the first statement without loss of generality we can assume that  $b > 0$ , for otherwise,  $|\Delta| = 1$  and  $\mathcal{Y} \uparrow L = \mathcal{Y}$ . Besides, by Lemma 3.1 we have  $gb(\mathcal{Y}_0) \leq gb(\mathcal{Y})$  where  $\mathcal{Y}_0$  is the coherent configuration defined in Lemma 7.3 with arbitrarily chosen point  $\gamma_0$ . Thus by Theorem 5.2 the coherent configuration  $\mathcal{Y}_0 \wr \text{Inv}(L)$  has a proper generalized base  $\Pi_0$  of size

$$b_0 \leq \max\{gb(\mathcal{Y}_0), b\} \leq \max\{gb(\mathcal{Y}), b\}.$$

By Lemma 7.3 this implies that the coherent configuration  $\mathcal{X}_0 = (\mathcal{X}_\alpha)_{\alpha r}$  with  $\alpha = \alpha(\gamma_0)$ , has a generalized base  $\Pi$  of size  $b_0$  that contains an element  $\Lambda_0$  such that

$$(31) \quad 0 < |\Lambda_0 \cap \Gamma_i| < |\Gamma_i| \quad \text{for all } i \in \Delta,$$

where the sets  $\Gamma_i$  are defined in (27). By the second statement of Lemma 7.2 the set  $\Pi$  is a generalized base of the coherent configuration  $\mathcal{X}_\alpha$ . Set  $\Phi$  be the fiber of  $\text{Fis}(\mathcal{X}, \Pi)$  that contains  $\alpha$ . Then it suffices to verify that  $\Phi = \{\alpha\}$  (indeed, in this case  $\text{Fis}(\mathcal{X}, \Pi) \geq \text{Fis}(\mathcal{X}_\alpha, \Pi)$  and we are done). To do this suppose that  $\beta \in \Phi$ . Then since  $\alpha \in \Phi$ ,  $\Lambda_0 \subset \alpha r$  and  $\Lambda_0$  is the union of fibers of  $\text{Fis}(\mathcal{X}, \Pi)$ , we have

$$(32) \quad \Lambda_0 \subset \beta r.$$

Then obviously  $d(\alpha, \beta) \leq 2$ . So without loss of generality we can assume that

$$(33) \quad \alpha_1 = \gamma \neq \beta_1 \quad \text{and} \quad \alpha_3 = \gamma = \beta_3$$

(since  $L$  is a transitive group of odd order, we can assume that  $m \geq 3$ ). However, by (31) there exists a point  $\delta \in \Lambda_0 \cap \Gamma_3$ . Then by (33) we have  $d(\delta, \beta) \geq 2$ . So  $\delta \notin \beta r$  which contradicts (32).

In the proof of the second statement we keep the notations of the previous paragraph. Since the coherent configuration  $\mathcal{Y}$  is not complete, we can choose the point  $\gamma_0 \in \Gamma$  so that there is a base of  $\mathcal{Y}$  of size  $b(\mathcal{Y})$  that contains  $\gamma_0$ . Then by Lemma 3.1 the coherent configuration  $\mathcal{Y}_0$  has a base of size at most  $b(\mathcal{Y}) - 1$ . So by Theorem 5.3 the coherent configuration  $\mathcal{Y}_0 \wr \text{Inv}(L)$  has a thin generalized base of size

$$(34) \quad b_0 \leq (b(\mathcal{Y}) - 1) + \max\{0, b - \lceil (b(\mathcal{Y}) - 1)/2 \rceil\}.$$

By Lemma 7.3 this implies that the coherent configuration  $\mathcal{X}_0$  has a generalized base  $\Pi$  of size  $b_0$  such that  $|\Lambda \cap \Gamma_i| \leq 1$  for all  $\Lambda \in \Pi$ . By Lemma 7.2 any such  $\Lambda$  is of the form  $\rho(\beta)$  for uniquely determined point  $\beta = \beta(\Lambda)$  in  $\Omega$ . Set

$$B_0 = \{\beta(\Lambda) : \Lambda \in \Pi\}.$$

Then  $\text{Fis}(\mathcal{X}_\alpha, B_0) \geq \text{Fis}(\mathcal{X}_\alpha, \Pi)$  because  $\rho(\beta)$  is a union of fibers of the coherent configuration  $\mathcal{X}_{\alpha, \beta}$  for all  $\beta \in B_0$ . Since  $\Pi$  is a generalized base  $\mathcal{X}_0$ , the second statement of Lemma 7.2 this implies that  $B_0$  is a base of the coherent configuration  $\mathcal{X}_\alpha$ . Thus the set  $B = B_0 \cup \{\alpha\}$  is a base of  $\mathcal{X}$ . Moreover,  $|B| = |\Pi| + 1 = b_0 + 1$  and the required statement follows from (34).  $\blacksquare$

## 8. INDISTINGUISHING NUMBER AND BASE NUMBER

Let  $\mathcal{X} = (\Omega, S)$  be a scheme. For any two points  $\alpha, \beta \in \Omega$  denote by  $\Omega_{\alpha, \beta}$  the set of all  $\gamma \in \Omega$  such that  $r(\alpha, \gamma) = r(\beta, \gamma)$ . Then

$$(35) \quad |\Omega_{\alpha, \beta}| = \sum_{t \in S} c_{tt}^s$$

where  $s = r(\alpha, \beta)$ . It follows that this number does not depend on the choice of  $(\alpha, \beta) \in s$  and is denoted by  $c(s)$ ; in [18] it was called the *indistinguishing number* of  $s$ . The maximal indistinguishing number of a non-reflexive basis relation of  $\mathcal{X}$  is denoted by  $c = c(\mathcal{X})$ . It is easily seen that  $c(\mathcal{X}) \geq 0$  and the equality is attained if and only if the scheme  $\mathcal{X}$  is regular.

The number  $n - c$  where  $n = |\Omega|$ , was called in paper [2] the distinguishing number of the coherent configuration  $\mathcal{X}$ . It was proved there that if  $\mathcal{X}$  is primitive and  $|S| \geq 3$ , then  $b(\mathcal{X}) \leq 4\sqrt{n} \log n$ . In the following theorem we are interested in the base number when  $\mathcal{X}$  is not necessarily primitive and  $c$  is rather small.

**Theorem 8.1.** *Let  $\mathcal{X}$  be a scheme such that  $4c(m-1) < n$  where  $m = n_{\max}$ . Then the coherent configuration  $\mathcal{X}_\alpha$  is 1-regular for any  $\alpha \in \Omega$ . In particular,  $b(\mathcal{X}) \leq 2$ .*

**Proof.** Without loss of generality we can assume that  $m \geq 2$  (otherwise the scheme  $\mathcal{X}$  is regular, and the statement is obvious). Let  $\alpha \in \Omega$  and  $r \in S$ . Given  $\beta \in \Omega$  denote by  $\Omega_\beta$  the set of all pairs  $(\delta, \gamma) \in \alpha r \times \alpha r$  such that  $\delta \neq \gamma$  and  $\beta \in \Omega_{\delta, \gamma}$ . Then it is easily seen that

$$|\Omega_\beta| = \sum_{s \in S} c_{rs}^t (c_{rs}^t - 1) = \sum_{s \notin r \circ t} c_{rs}^t = \sum_{s \in S} c_{rs}^t - |r \circ t| = n_r - |r \circ t|$$

where  $t = r(\alpha, \beta)$  and  $r \circ t = \{s \in r^*t : c_{rs}^t = 1\}$ . This implies that if  $\beta \in \alpha S'_r$  where  $S'_r$  is the set of all  $t' \in S$  with  $|r \circ t'| < n_r/2$ , then the set  $\Omega_\beta$  has at least  $n_r/2$  elements. Therefore

$$(36) \quad |\alpha S'_r| \cdot \frac{n_r}{2} \leq \sum_{\beta \in \alpha S'_r} |\Omega_\beta| \leq |T|$$

where  $T$  is the union of all  $\Omega_\beta$  with  $\beta \in \alpha S'_r$ . However, for each pair  $(\delta, \gamma) \in \alpha r \times \alpha r$  with  $\delta \neq \gamma$  there are at most  $c$  points  $\beta$  such that  $(\delta, \gamma) \in \Omega_\beta$ . So the set  $T$  has at most  $n_r(n_r - 1)c$  elements. By (36) and the lemma hypothesis this implies that  $|\alpha S'_r| \leq 2(n_r - 1)c \leq 2(m - 1)c < n/2$ . Thus

$$(37) \quad |\alpha S_r| = n - |\alpha S'_r| > \frac{n}{2}$$

where  $S_r = \{t \in S : |r \circ t| > n_r/2\}$  is the complement to  $S'_r$ .

To complete the proof we will show that any  $\beta \in \Omega$  for which the relation  $r = r(\alpha, \beta)$  is of valency  $m$ , is a regular point of the coherent configuration  $\mathcal{X}_\alpha$ , i.e. that

$$(38) \quad \beta r_{\mathcal{X}_\alpha}(\beta, \gamma) = \{\gamma\}$$

for all  $\gamma \in \Omega$ . To do this set  $u = r(\alpha, \gamma)$ . Then inequality (37) implies that  $|\alpha S_r| > n/2$  and  $|\alpha S_u| > n/2$ . Therefore the sets  $S_r$  and  $S_u$  contain a common relation, say  $v$ . It follows that neither  $r \circ v$  nor  $u \circ v$  is empty; take  $s_\beta \in r \circ v$  and  $s_\gamma \in u \circ v$ . Then by the definition of  $\circ$  one can find points  $\beta'$  and  $\gamma'$  in  $\alpha v$  such that

$$(39) \quad \beta' s_\beta^* \cap \alpha r = \{\beta\} \quad \text{and} \quad \gamma' s_\gamma^* \cap \alpha u = \{\gamma\}.$$

Moreover, we have  $|r \circ v| > n_r/2$  because  $v \in S_r$ . Therefore one can find two relations  $t_\beta$  and  $t_\gamma$  in  $r \circ v$  such that

$$(40) \quad \beta' t_\beta^* \cap \alpha r = \{\delta\} = \gamma' t_\gamma^* \cap \alpha r$$

for some point  $\delta \in \alpha r$ . The obtained configuration is represented at Fig. 1. By Lemma 2.2 the set  $(S_\alpha)^\cup$  contains the relations

$$a_1 = (s_\beta)_{\alpha r, \alpha v}, \quad a_2 = (t_\beta^*)_{\alpha v, \alpha r}, \quad a_3 = (t_\gamma)_{\alpha r, \alpha v}, \quad a_4 = (s_\gamma^*)_{\alpha v, \alpha u},$$

and hence the relation  $a = a_1 \cdot a_2 \cdot a_3 \cdot a_4$ . On the other hand, since  $n_r = m$  and  $v \in S_r$ , from (4) it follows that  $n_v = m$ . This implies that  $s_\beta^*, t_\beta^*, t_\gamma^* \in S(v, r)$ . Therefore due to (39) and (40) we obtain that

$$(41) \quad \beta a_1 = \{\beta'\}, \quad \beta' a_2 = \{\delta\}, \quad \delta a_3 = \{\gamma'\}, \quad \gamma' a_4 = \{\gamma\}.$$

Thus  $\beta r_{\mathcal{X}_\alpha}(\beta, \gamma) \subset \beta a = \{\gamma\}$  whence (38) follows.  $\blacksquare$

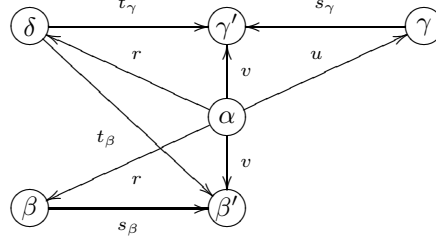


FIGURE 1.

**Corollary 8.2.** *Let  $G \leq \text{AGL}(\Omega)$  be an affine group acting on a linear space  $\Omega$  over a finite field, and  $K$  a one point stabilizer of  $G$ . Suppose that*

$$(42) \quad 4(k-1)\text{Fix}(K) < n.$$

where  $n = |\Omega|$  and  $k = |K|$ . Then  $b(\text{Inv}(G)) \leq 2$ . In particular, this is always true whenever  $4k(k-1)f < n$  where  $f = \text{fix}(K)$ .

**Proof.** Set  $\mathcal{X} = \text{Inv}(G)$ . Choose two points  $\alpha$  and  $\beta$  such that  $c = c(s) = |\Omega_{\alpha,\beta}|$  where  $s = r(\alpha, \beta)$ . Then any point in  $\Omega_{\alpha,\beta}$  is a fixed point of a permutation from the set  $G_{\alpha \rightarrow \beta} = \{a \in G : \alpha^a = \beta\}$ . Therefore

$$(43) \quad c = |\Omega_{\alpha,\beta}| \leq \text{Fix}(G_{\alpha \rightarrow \beta}).$$

On the other hand, any  $a \in G_{\alpha \rightarrow \beta}$  is an affine mapping on  $\Omega$ , say  $x \mapsto hx + b$  where  $h$  is the matrix from  $G_\alpha$  and  $b = \beta - \alpha$  is a vector. Therefore the numbers  $\text{fix}(a)$  and  $\text{fix}(h)$  are equal respectively to the numbers of solutions of linear equation systems  $(h - e)x = b$  and  $(h - e)x = 0$  where  $e$  is the identity matrix. When  $0 < \text{fix}(a) \leq n$ , the latter numbers are equal. Therefore the right-hand side of (43) coincides with  $\text{Fix}(G_\alpha)$ . Thus since  $G_\alpha$  and  $K$  are conjugate in  $G$ , we obtain from (43) that

$$(44) \quad c \leq \text{Fix}(K).$$

Next, since  $\mathcal{X}$  is the scheme of the transitive group  $G$ , we have  $n_s \leq k$  for all  $s \in S$ . Therefore it follows from (44) and (42) that

$$4(m-1)c \leq 4(k-1)\text{Fix}(K) < n.$$

Thus the first statement of the theorem follows from Theorem 8.1. The second statement also follows from the above inequality because  $\text{fix}(K) \leq (k-1)f$ . ■

## 9. BASE OF LINEARLY PRIMITIVE ANTISYMMETRIC SCHEME

In this section we will prove that, in fact, the base number of a linearly primitive antisymmetric scheme coincides with the base number of its automorphism group. However, our proof do not use the fact that the latter number is at most 3.

**Theorem 9.1.** *The base number of a linearly primitive antisymmetric scheme is at most 3 and the equality is attained only for cyclotomic schemes over a finite field.*

In what follows we fix an affine group  $G \leq \text{AGL}(d, p)$  such that the scheme  $\mathcal{X} = \text{Inv}(G)$  is antisymmetric and linearly primitive. Then by Theorem 2.3 the zero stabilizer in  $G$  is an irreducible primitive group  $K \leq \text{GL}(d, p)$  of odd order.

For this group we keep the notation of Theorem 4.1. In the following three lemmas we will verify that

$$(45) \quad e > 1 \Rightarrow b(\mathcal{X}) \leq 2.$$

In each of these lemmas we will subsequently exclude the values of  $e$  for which the implication could be violate, by means of Lemma 3.4 and Corollary 8.2.

**Lemma 9.2.** *The implication (45) holds unless the quadruple  $(e, a, d, p)$  is one of the following:*

- (E1)  $(e, a, d) = (9, 1, 9)$  and  $p \in \{7, 13, 19, 31, 37, 43\}$ ,
- (E2)  $(e, a, d, p) = (5, 4, 20, 3), (5, 3, 15, 11)$  or  $(5, 2, 10, 11)$ ,
- (E3)  $(e, a, d) = (5, 1, 5)$  and  $p \in \{11, \dots, 5591\}$ ,  $p \equiv 1 \pmod{5}$ .

**Proof.** Suppose that the parameters  $e, a, d, p$  of the group  $K$  do not form a quadruple from the lemma statement. Then by Corollary 8.2 it suffices to verify that  $p^d > 4k^2f$  where  $k = |K|$  and  $f = \text{fix}(K)$ . However,  $f \leq p^{\lfloor 4d/9 \rfloor}$  by Theorem 4.3. Therefore due to (17) the required inequality is a consequence of the following one:

$$(46) \quad p^d > 4 \cdot (u_{a,p} \cdot e^2 \cdot s_e \cdot a_0)^2 \cdot p^{\lfloor 4d/9 \rfloor}.$$

Here  $ae \leq d$  by statement (P4) of Theorem 4.1. Therefore  $a_0 \leq a \leq d/e$  and  $2u_{a,p} \leq p^a \leq p^{d/e}$ . Besides, by the second and the third statements of Lemma 4.2 we have  $s_e \leq e^2/2$ . Consequently,  $4 \cdot u_{a,p} \cdot s_e \cdot a_0 \leq p^{d/e} \cdot d \cdot e$ . Thus to check inequality (46) it suffices to verify that

$$(47) \quad 4 \cdot p^{d - \lfloor 4d/9 \rfloor} > p^{2d/e} \cdot d^2 e^6.$$

A direct computation shows that  $4 \cdot 3^{14d/27} > d^8$  for all  $d \geq 54$ . Therefore for all integers  $e \geq 54$  and all primes  $p \geq 3$  the inequality

$$4 \cdot p^{d - \lfloor 4d/9 \rfloor - 2d/e} \geq 4 \cdot 3^{(5/9 - 2/54)d} = 4 \cdot 3^{14d/27} > d^8 \geq d^2 e^6.$$

holds for all  $d \geq e$ . This proves the required statement for all  $e \geq 54$ .

Denote by  $d(e, p)$  the minimal positive integer  $d$  for which inequality (47) holds for a fixed  $e$  and  $p$ , and by  $p(a, e)$  the minimal element in the set  $P(a, e)$  of all odd primes  $q$  such that each prime divisor of  $e$  divides  $q^a - 1$ . Then by statements (P2) and (P4) of Theorem 4.1 and by Lemma 4.2 without loss of generality we can assume that

- (C1)  $e \in \{5, \dots, 53\}$  is an odd integer other than 7,
- (C2) when  $e$  is fixed,  $a \in \{1, \dots, \lfloor d_0/e \rfloor\}$  where  $d_0 = d(e, 3)$ ,
- (C3) when  $e$  and  $a$  are fixed,  $p \in P(a, e)$ .

For each  $e$  satisfying (C1) we list in the Table 1 below the values of the function  $d_3 = d(e, 3)$  (the second row), the possible values for the integer  $a$  (the third row), and for a fixed  $a$  also the values of the functions  $p_a = p(a, e)$  and  $d_p = d(e, p_a)$  (the fourth and the fifth rows respectively).<sup>7</sup>

From the above definitions it follows that for a fixed pair  $(e, a)$  satisfying conditions (C1) and (C2) and such that  $d(e, p_a) \leq ea$ , inequality (47) holds for all  $d \geq ae$  and  $p \in P(a, e)$ . This enables us to find all the pairs for which inequality (47) does not hold for at least one  $p \in P(a, e)$  (the corresponding values of  $a$  in Table 1 are written in bold script):

<sup>7</sup> When  $e = 5$ , we have  $p_a = 3$  and  $d(e, p_a) = 55$  for  $a \equiv 0 \pmod{4}$ , and  $p_a = 11$  and  $d(e, p_a) = 43$  otherwise.

TABLE 1.

$e$	53	51	49	47	45	43	41	39	37	35	33	31	29	27	25	
$d_3$	54	54	53	53	53	53	52	52	52	51	51	51	50	50	50	
$a$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1,2	
$p_a$	107	103	29	283	31	173	83	79	149	71	67	311	59	7	11	
$d_p$	12	12	16	10	16	10	12	12	10	12	12	9	12	27	22	
$e$	23	21		19		17		15		13		11		9		5
$d_3$	49	49		49		49		49		50		51		55		103
$a$	1,2	1	2	1	2	1	2	1,3	2	1,2	<b>3</b>	1,2,3,4	<b>1,3,5</b>	<b>2,4,6</b>	1..20	
$p_a$	47	43	13	191	37	103	67	31	11	53	3	23	7	5	3,11	
$d_p$	13	13	20	9	14	10	12	14	21	12	50	16	29	36	55,43	

- $(e, a) = (13, 3)$  or  $(11, 1)$ ,
- $e = 9$  and  $a \in \{1, 2, 3\}$ ,
- $e = 5$  and  $a \in \{1, \dots, 8\}$ .

For each of these pairs we have to check inequality (46) for all positive integers  $d \leq d(e, 3)$  which is a multiple of  $ae$ . The *available* triples  $(e, a, d)$ , i.e. those that are obtained in this way, are listed in the first three rows of Table 2 below.<sup>8</sup> In the fourth and the fifth rows of this table we give respectively the values  $p = p(a, e)$  and  $q = q(e, a, d)$  where the latter number is equal to the minimal prime in  $P(a, e)$  for which

$$(48) \quad q^{d - \lfloor 4d/9 \rfloor} > 4 \cdot ((q^a - 1)/t_a \cdot e^2 \cdot s_e \cdot a_0)^2;$$

Here the integer  $t_a$  is defined as follows: if  $a$  is odd, then  $t_a = 2$ , otherwise  $t_a = 2^{t+2}$  where  $t$  is the maximal positive integer such that  $2^t$  divides  $a$ . Then obviously  $t_a$  divides  $p^a - 1$  for any odd prime  $p$ , and hence  $u_{a,p} \leq (p^a - 1)/t_a$ . Thus the required inequality (46) follows from (48). In the computation of  $q$  we used the values of  $s_e$  (and in cases  $(e, a, d) = (5, 3, 15)$  and  $(5, 5, 25)$  also the equality  $|K : F| = 3$ ) given in the second statement of Lemma 4.2.

TABLE 2.

e	13	11	9	9	9	9	5	5	5
a	3	1	1	1	3	2	4	4	8
d	39	11	<b>9</b>	18	27	18	<b>20</b>	40	40
p	3	23	7	7	7	5	3	3	3
q	3	23	61	5	5	5	7	3	3
e	5	5	5	5	5	5	5	5	5
a	1	1	2	2	3	3	5	6	7
d	<b>5</b>	10	<b>10</b>	20	<b>15</b>	30	25	30	35
p	11	11	11	11	11	11	11	11	11
q	5641	11	19	3	31	3	11	7	11

<sup>8</sup>We did not cited in the table some available triples, like  $(e, a, d)$  with  $d \geq 22$  and  $(e, a) = (11, 1)$ , because if the inequality (48) holds for some  $d$ , then it holds also for largest  $d$ 's.

It follows from the definition of  $q$  that if  $(e, a, d)$  is one of the available triples and  $q \leq p$ , then inequality (46) holds for all  $p \in P(a, e)$ . The remaining 5 triples are the following:  $(9, 1, 9)$ ,  $(5, 4, 20)$ ,  $(5, 1, 5)$ ,  $(5, 2, 10)$  and  $(5, 3, 15)$  (the corresponding values of  $d$  in Table 2 are written in bold script). For any of them the inequality (46) does not hold only for those quadruples  $(e, a, d, p)$  in which

$$p \in P(a, e) \cap \{1, \dots, q-1\}.$$

A straightforward check shows that these quadruples are exactly those listed in the lemma statement.  $\blacksquare$

**Lemma 9.3.** *The implication (45) holds unless  $(e, a, d) = (9, 1, 9)$  and  $p \in \{7, 19\}$ , or  $(e, a, d) = (5, 1, 5)$  and  $p \in \{11, 31, 41, 61, 71, 101, 151, 181, 271\}$ .*

**Proof.** Given a prime  $q$  denote by  $k_q$  the number of all non-identity elements  $g \in K$  the order of which is a power of  $q$ ; the maximum of  $\text{fix}(g)$  over all these elements  $g$  is denoted by  $f_q$ . Clearly, this maximum is achieved on the elements of order  $q$ . The number  $f_{q'}$  is defined in a similar way: the maximum is taken over all non-identity elements  $g \in K$  the order of which is not a power of  $q$ . Then it is easily seen that  $\text{Fix}(K) \leq k_q f_q + (k - k_q) f_{q'}$ . So by Corollary 8.2 it suffices to prove that the inequality

$$(49) \quad p^d > 4(k-1)(k_q f_q + (k - k_q) f_{q'}).$$

holds for an appropriate prime divisor  $q$  of  $k = |K|$ . By Lemma 9.2 it suffices to check this inequality only for those groups  $K$  the parameters  $(e, a, d, p)$  of which are listed in the statement of this lemma.

Let  $(e, a, d) = (9, 1, 9)$  and  $p \in \{13, 31, 37, 43\}$ . Then from Theorem 4.1 it follows that  $K = A$ ,  $|F : U| = 3^4$  and  $U$  is a central subgroup of  $K$ . Besides, by Lemma 4.2 we also have  $|A : F| = s_e = 5$ . Thus

$$k = |F| \cdot 5 \quad \text{divides} \quad p_0 := \frac{p-1}{2} \cdot 3^4 \cdot 5.$$

It follows that the order of a Sylow 5-subgroup of  $K$  is 5 or 25 depending on whether  $p \in \{13, 37, 43\}$  or  $p = 31$ ; in the former case  $k_5 = 4 \cdot 3^4$ , whereas in the latter one  $k_5 = 20 \cdot 3^4$ . Moreover, in any case one can easily deduce from Lemma 4.3 that  $f_5 \leq p^{\lfloor d/5 \rfloor} = p$  and  $f_{5'} \leq p^{d/3} = p^3$ . Thus

$$k_5 f_5 + (k - k_5) f_{5'} \leq k_5 p + (p_0 - k_5) p^3.$$

A straightforward computation shows that the right-hand side of this inequality is less than  $p^9/4(k-1)$  for  $p \in \{13, 31, 37, 43\}$ . This proves required inequality (49) in our case.

A similar argument works when the quadruple  $(e, a, d, p)$  is equal to  $(5, 4, 20, 3)$ ,  $(5, 3, 15, 11)$  or  $(5, 2, 10, 11)$ . In all these cases  $|K : F| = 3$  by Lemma 4.2. From now on we always assume that the group  $U$  is the maximal odd subgroup of the multiplicative subgroup of  $\text{Span}(U) = \text{GF}(p^a)$  (the base of a scheme is not decreased under taking a fusion). Then from Theorem 4.1 and Lemma 4.3 it follows that  $k = 75 \cdot u$  with  $u = |U|$ ,  $f_3 \leq p^{\lfloor 4d/9 \rfloor}$  and

- if  $(e, a, d, p) = (5, 4, 20, 3)$ , then  $u = 5$ ,  $k_3 = 5^2 \cdot 2$  and  $f_{3'} \leq p^4$ ,
- if  $(e, a, d, p) = (5, 3, 15, 11)$ , then  $u = 35 \cdot 19$ ,  $k_3 \leq 7 \cdot 19 \cdot 5^2 \cdot 2$  and  $f_{3'} \leq p^3$ ,
- if  $(e, a, d, p) = (5, 2, 10, 11)$ , then  $u = 15$ ,  $k_3 = 5^2 \cdot 6$  and  $f_{3'} \leq p^2$ .

In all these cases inequality (49) with  $q = 3$  follows by a straightforward computation (for  $d = 15$  we have even more strong inequality in which the summand  $(k - k_q)f_{q'}$  is replaced by  $kf_{q'}$ ).

Let  $(e, a, d) = (5, 1, 5)$  and  $p$  belongs to the set  $\mathcal{P}_0$  of all primes  $q \leq 5591$  such that  $q \equiv 1 \pmod{5}$ . In this case as before we have:  $K = A$ ,  $|A : F| = 3$ ,  $|F : U| = 5^2$  and  $U$  is a central subgroup of  $K$  (Theorem 4.1 and Lemma 4.2). Thus

$$(50) \quad k = |F| \cdot 3 \quad \text{divides} \quad \frac{p-1}{2} \cdot 5^2 \cdot 3.$$

A straightforward computation for all  $p \in \mathcal{P}_0$  shows that the order of a Sylow 3-subgroup of the group  $K$  equals to  $3^{t_p}$  where  $1 \leq t_p \leq 6$ . By Lemma 4.3 we have

$$(51) \quad k_3 = 25(3^{t_p} - 3^{t_p-1}), \quad f_3 \leq p^{\lfloor 4d/9 \rfloor} = p^2, \quad f_{3'} \leq p^{\lfloor d/3 \rfloor} = p$$

Next, given a positive integer  $t$  denote by  $p_0 = p_0(t)$  the minimal prime in  $\mathcal{P}_0$  for which  $3^{t-1}$  divides  $(p_0 - 1)/2$ , and by  $p_1 = p_1(t)$  the maximal real root of the polynomial

$$g_t(x) = x^5 - 4 \cdot (x' - 1) \cdot (t'x^2 + (x' - t')x),$$

where  $x' = 75(x - 1)/2$  and  $t' = 25(3^t - 3^{t-1})$ . When  $t = t_p$  and  $x = p$ , we obtain from (50) and (51) that  $x' \geq k$  and  $t' = k_3$ . Therefore

$$p^5 \geq p^5 - g_{t_p}(p) \geq 4 \cdot (k - 1) \cdot (k_3 f_3 + (k - k_3) f_{3'}).$$

On the other hand, it is easily seen that  $g_t(p) > 0$  for all  $t \geq 1$  and all  $p > p_1(t)$ . Thus inequality (49) does not hold only if  $t \in \{1, \dots, 6\}$  and  $p \in \mathcal{P}_0$  is such that  $p_0(t) \leq p \leq p_1(t)$ . In the Table 3 we present computed values of the functions  $p_0(t)$  and  $p_1(t)$ . It follows that the required inequality does not hold only if  $(p, t)$  is one of the following pairs:  $(271, 4)$ ,  $(181, 3)$ ,  $\{31, 61, 151\} \times \{2\}$  and  $\{11, 41, 71, 101\} \times \{1\}$ . Thus the proof in this case is completely done.  $\blacksquare$

TABLE 3.

$t$	1	2	3	4	5	6
$p_0$	11	31	181	271	811	4861
$\lfloor p_1 \rfloor$	113	166	269	455	782	1351

**Lemma 9.4.** *The implication (45) holds for all  $e > 1$ .*

**Proof.** We recall that the set  $\Omega$  is identified with a  $d$ -dimensional linear space over a field  $\text{GF}(p)$  and the group  $G$  contains the translation group of  $\Omega$ . Denote by  $\alpha$  the zero vector of  $\Omega$ . Then given  $r \in S$  and  $\beta \in \alpha r$  the intersection number  $c_{ts}^r$  is equal to the number of all  $\gamma \in \alpha t$  for which  $r(\gamma, \beta) = s$ . Besides, it is easily seen that  $r(\gamma, \beta) = r(\gamma', \beta)$  if and only if  $\gamma - \beta \in (\gamma' - \beta)^K$  for all  $\gamma'$ . Thus

$$(52) \quad c_{ts}^r = |\Delta_\beta(t)|$$

where  $\Delta_\beta(t)$  the set of all sets  $(\gamma - \beta)^K \cap (\alpha t - \beta)$  with  $\gamma \in \alpha t$  and  $\alpha t - \beta$  is the set of all vectors  $\gamma' - \beta$  with  $\gamma' \in \alpha t$ . Then using (52) for  $t = r$  and  $t = r^*$  we can compute the numbers  $r \circ_2 r$  and  $|r \circ_2 r^*|$  defined before Lemma 3.4 as follows:

$$|r \circ_2 r| = |\{\Delta \in \Delta_\beta(r) : |\Delta| \leq 2\}| =: a_r$$



and

$$|r \circ_2 r^*| = |\{\Delta \in \Delta_\beta(r^*) : |\Delta| \leq 2\}| =: b_r$$

(in the second case we used equalities  $c_{rs}^{r^*} = c_{s^*r^*}^r = c_{r^*s^*}^r$  that follow from (4)). Our goal is to find a relation  $r \in S$  such that

$$(53) \quad a_r + b_r > 2n_r/3.$$

Then Lemma 3.4 implies that  $b(\mathcal{X}) \leq 2$ , and we are done. To find such  $r$  we can assume that  $(e, a, d, p)$  is one of the quadruples listed in the statement of Lemma 9.3.

Suppose first that  $(d, p) = (9, 7)$ . Denote by  $E$  an extraspecial group of order  $3^5$  and exponent 3. Then  $K = A$  is isomorphic to a semidirect product  $K_0 = E.5$  in which the group of order 5 acts irreducibly on  $E/Z(E)$ . By means of GAP we found that  $K_0$  is uniquely determined up to isomorphism (there is a unique non-nilpotent group of order  $3^5 \cdot 5$  with a nonabelian Sylow 3-subgroup). Moreover, up to equivalence there are exactly two classes of irreducible  $d$ -dimensional  $K_0$ -modules over  $\text{GF}(p)$ . For both of them we constructed in GAP generators for the corresponding primitive subgroup of  $\text{GL}(d, p)$  isomorphic to  $K$  (see Section 4). Then we fixed a standard linear base  $\{e_1, \dots, e_d\}$  in  $\text{GF}(p)^d$  and took

$$\beta = e_1 + e_2 + e_5 \quad \text{and} \quad r = r(\alpha, \beta).$$

A straightforward computation shows that in both cases  $n_r = |\beta^K| = |K| = 1215$ ,  $a_r = 1035$  and  $b_r \geq 754$ . Therefore inequality (53) do hold and we are done.

The computation in each of the remaining case is essentially the same as in the above case  $(d, p) = (9, 7)$ . The minor differences are the following. In the case  $(d, p) = (9, 19)$  we have  $K = \langle K_0, \xi_p I_d \rangle$  where  $I_d$  is the identity matrix in  $\text{GL}(d, p)$ , and  $\xi_p \in \text{GF}(p)$  is a generator of the maximal multiplicative  $2'$ -subgroup in  $\text{GF}(p)$  (in our case, the subgroup of order 9). In the case  $(5, p)$  the group  $E$  is an extraspecial group of order  $5^3$  and exponent 5,  $K_0$  is a semidirect product  $E.3$  in which the group of order 3 acts irreducibly on  $E/Z(E)$ , and  $K = \langle K_0, \xi_p I_d \rangle$ . The computation results cited in the Table 4 below show that inequality (53) holds

TABLE 4.

	(9, 9, 1)		(5, 5, 1)								
$p$	7	19	11	31	41	61	71	101	151	181	271
$\beta$	$e_1 + e_2 + e_5$		$e_1 + e_2$								
$n_r$	1215	3645	375	1125	375	1125	2625	1875	5625	3375	10125
$a_r$	1035	3483	199	987	361	1061	2413	1755	5221	3199	9421
$b_r$	754	1697	99	469	160	526	1181	853	2585	1563	4685
$N$	2	2	4	12	4	12	4	4	12	12	12

in all the cases (in the last row in the table we gives the number of the classes of irreducible  $d$ -dimensional  $K_0$ -modules over  $\text{GF}(p)$ ; the values  $a_r$  and  $b_r$  correspond to the  $K_0$ -module with minimal sum  $a_r + b_r$ ). ■

**Proof of Theorem 9.1.** Let  $\mathcal{X} = \text{Inv}(G)$  where  $G \leq \text{AGL}(d, p)$  is an affine group with primitive zero stabilizer  $K \leq \text{GL}(d, p)$  of odd order,  $p$  is an odd prime. Then from Lemmas 9.4 and 4.2 it follows that  $K$  is contained in the unique Hall  $2'$ -subgroup  $K^*$  of the group  $\text{GL}(1, p^d)$ . It is easily seen that  $\text{Orb}(K^*, \Omega) = \text{Orb}(K', \Omega)$

where  $K'$  is the maximal odd order multiplicative group of the field  $\mathbb{F} = \text{GF}(p^d)$ . Thus

$$(54) \quad \mathcal{X} = \text{Inv}(G) \geq \text{Inv}(G^*) = \text{Inv}(G') =: \mathcal{X}'$$

where  $G^* = AK^*$  and  $G' = AK'$  with  $A$  being the translation group of the linear space  $\Omega$ . However,  $\mathcal{X}'$  is a cyclotomic scheme over the field  $\mathbb{F}$ . Therefore  $b(\mathcal{X}') \leq 3$  by Theorem 3.2. Thus by (54) we conclude that  $b(\mathcal{X}) \leq b(\mathcal{X}') \leq 3$  which completes the proof.  $\blacksquare$

## 10. THE PROOF OF THEOREMS 1.1, 1.2 AND 1.4

**10.1. Proof of Theorem 1.2.** We argue by induction on the degree of a schurian antisymmetric coherent configuration  $\mathcal{X} = \text{Inv}(G)$ . By the first inequality in (9), we can assume that it is homogeneous. Suppose that  $\mathcal{X}$  is imprimitive. Then there is a nontrivial equivalence relation  $e \in S^\cup$ . The schemes  $\mathcal{X}_1 = \mathcal{X}_\Gamma$  where  $\Gamma \in \Omega/e$ , and  $\mathcal{X}_2 = \mathcal{X}_{\Omega/e}$  are obviously antisymmetric and schurian. Moreover,  $\mathcal{X}$  is isomorphic to a fission of the scheme  $\mathcal{X}_1 \wr \mathcal{X}_2$ . By Corollary 5.2 this implies that

$$gb(\mathcal{X}) \leq gb(\mathcal{X}_1 \wr \mathcal{X}_2) \leq \max\{gb(\mathcal{X}_1), gb(\mathcal{X}_2)\}$$

and we are done by induction.

Suppose that the scheme  $\mathcal{X}$  is primitive. Then by Theorem 2.3 it is either linearly imprimitive or linearly primitive. In the former case  $\mathcal{X}$  has a nontrivial fusion isomorphic to  $\mathcal{Y} \uparrow L$  where  $\mathcal{Y}$  is a linearly primitive antisymmetric scheme and  $L$  is a transitive group of odd order (Theorem 6.1). Thus by induction and Theorem 7.1 we have

$$gb(\mathcal{X}) \leq gb(\mathcal{Y} \uparrow L) \leq \max\{gb(\mathcal{Y}), gb(\text{Inv}(L))\} \leq 1.$$

To complete the proof suppose that  $\mathcal{X}$  is linearly primitive. If it is cyclotomic, then we are done by Theorem 3.2. Otherwise by Theorem 9.1 it has a base  $B = \{\alpha, \beta\}$  where  $\alpha$  and  $\beta$  are two (possibly equal) points in  $\Omega$ . In this case

$$\text{Fis}(\mathcal{X}, \{B\}) = \mathcal{X}_{\alpha, \beta}$$

because the scheme  $\mathcal{X}$  is antisymmetric. Thus  $gb(\mathcal{X}) \leq 1$  and we are done.  $\blacksquare$

**10.2. Proof of Theorem 1.1.** By Theorem 2.3 the scheme  $\mathcal{X} = \text{Inv}(G)$  is either linearly imprimitive or linearly primitive. In the latter case we are done by Theorem 9.1. In the former case  $\mathcal{X}$  has a nontrivial fusion isomorphic to  $\mathcal{Y} \uparrow L$  where  $\mathcal{Y}$  is a linearly primitive antisymmetric scheme and  $L$  is a transitive group of odd order (Theorem 6.1). In particular, the scheme  $\mathcal{Y}$  is not complete. Besides,  $b(\mathcal{Y}) \leq 3$  by Theorem 9.1 and  $b := gb(\text{Inv}(L)) = 1$  by Theorem 1.2. This implies by Theorem 7.1 that

$$b(\mathcal{X}) \leq b(\mathcal{Y} \uparrow L) \leq b(\mathcal{Y}) + \max\{0, 1 - \lceil (b(\mathcal{Y}) - 1)/2 \rceil\}.$$

When  $b(\mathcal{Y}) = 1, 2, 3$ , the right-hand side of the above inequality is equal respectively to 2, 2, 3. In any case  $b(\mathcal{X}) \leq 3$ , and we are done.  $\blacksquare$

**10.3. Algorithm.** We will deduce Theorem 1.4 from Theorem 1.3 proved below. The algorithm constructed in the proof of the latter theorem is, in a sense, a combinatorial version of the Babai-Luks algorithm from [3]. The following statement to be used in the proof of Theorem 1.3 is a special case of Corollary 3.6 of that paper. In what follows we always assume that any permutation group on the input or output of an algorithm is given by a generator set of polynomial size in the degree of the group.

**Theorem 10.1.** *Let  $G \leq \text{Sym}(\Omega)$  be a solvable group of degree  $n$ . Then given a coherent configuration  $\mathcal{X}$  on  $\Omega$ , the group  $\text{Aut}(\mathcal{X}) \cap G$  can be found in time  $n^{O(1)}$ . ■*

To apply Theorem 10.1 we have to be able to construct the group  $G$ . This will be done by means of Corollary 3.6 and the following statement proved in [9, Lemma 3.4]. Below for permutation groups  $G_1, \dots, G_s$ ,  $s \geq 1$ , we define the group  $\text{Wr}(G_1, \dots, G_s)$  to be the iterated wreath product  $(\dots(G_1 \wr G_2) \wr \dots) \wr G_s$  in imprimitive action.

**Lemma 10.2.** *Let  $\mathcal{X}$  be a scheme and  $1_\Omega = e_0 \subset e_1 \subset \dots \subset e_s = \Omega^2$  a series of equivalence relations of  $\mathcal{X}$ . Suppose that for  $i = 1, \dots, s$  a permutation group  $G_i$  on a set  $\Delta_i$  and a family of bijections  $f_\Gamma : \Gamma \rightarrow \Delta_i$  where  $\Gamma \in \Omega_i$  with  $\Omega_i = \{\Gamma'/e_{i-1} : \Gamma' \in \Omega/e_i\}$ , form a majorant of  $\text{Aut}(\mathcal{X})$  with respect to the pair  $(e_{i-1}, e_i)$ . Then the mapping*

$$f : \Omega \rightarrow \prod_{i=1}^s \Delta_i, \quad \alpha \mapsto (\dots, f_i(\Gamma_{i-1}), \dots)$$

*is a bijection and  $\text{Aut}(\mathcal{X})^f \leq \text{Wr}(G_1, \dots, G_s)$  where  $f_i = f_{\Gamma_i}$  and  $\Gamma_{i-1}$  and  $\Gamma_i$  are respectively the classes of  $e_{i-1}$  and  $e_i$  containing  $\alpha$ . ■*

**Proof of Theorem 1.3.** To describe the algorithm we need the auxiliary procedure  $\text{Test}(\mathcal{X}, G)$  that given a coherent configuration  $\mathcal{X} = (\Omega, S)$  and a group  $G \leq \text{Sym}(\Omega)$  output  $G$  or empty set depending whether or not  $\mathcal{X} = \text{Inv}(G)$ . Since the latter equality exactly means that  $S = \text{Orb}(G, \Omega^2)$ , the procedure can be implemented in polynomial time in  $|\Omega|$  by means of a standard algorithm finding the orbits of a permutation group (see e.g. [20]).

### Schurity Recognition Algorithm

**Input:** an antisymmetric coherent configuration  $\mathcal{X}$ .

**Output:** the group  $\text{Aut}(\mathcal{X})$ , or  $\mathcal{X}$  is not schurian.

**Step 1.** If  $\mathcal{X}$  is not homogeneous, then recursively apply the algorithm to the coherent configuration  $\mathcal{X}_1 = \mathcal{X}_{\Delta_1}$  and  $\mathcal{X}_2 = \mathcal{X}_{\Delta_2}$  where  $\Delta_1$  is a fiber of  $\mathcal{X}$  and  $\Delta_2$  is its complement. If either  $\mathcal{X}_1$  or  $\mathcal{X}_2$  is not schurian, then so is  $\mathcal{X}$ , else output  $\text{Test}(\mathcal{X}, H)$  where  $H \leq \text{Sym}(\Omega)$  is the group found by the algorithm of Theorem 10.1 for  $G = \text{Aut}(\mathcal{X}_1) \times \text{Aut}(\mathcal{X}_2)$ .

**Step 2.** Find a maximal series of equivalence relations as in the hypothesis of Lemma 10.2. If there exist  $i \in \{1, \dots, s\}$  and  $\Gamma, \Gamma' \in \Omega_i$  such that

$$(55) \quad b(\mathcal{X}_\Gamma) > 3 \quad \text{or} \quad \text{Iso}(\mathcal{X}_\Gamma, \mathcal{X}_{\Gamma'}, \varphi_{\Gamma, \Gamma'}) = \emptyset$$

where  $\varphi_{\Gamma, \Gamma'}$  is the algebraic isomorphism (6) for  $\mathcal{X} = \mathcal{X}_{\Omega/e_{i-1}}$ , then  $\mathcal{X}$  is not schurian.

**Step 3.** By the algorithm of Corollary 3.6 find a majorant of  $\text{Aut}(\mathcal{X})$  with respect to  $(e_{i-1}, e_i)$ , say  $G_i \leq \text{Sym}(\Delta_i)$  and  $\{f_\Gamma\}_{\Gamma \in \Omega_i}$  for  $i = 1, \dots, s$ .

**Step 4.** Output  $\text{Test}(\mathcal{X}, H)$  where  $H = \text{Aut}(\mathcal{X}) \cap G^{f^{-1}}$  is the group found by the algorithm of Theorem 10.1 with  $G = \text{Wr}(G_1, \dots, G_s)$  and  $f$  as in Lemma 10.2. ■

To prove the correctness of the algorithm suppose first that the coherent configuration  $\mathcal{X}$  is not homogeneous. Then it is schurian only if so are the coherent configurations  $\mathcal{X}_1$  and  $\mathcal{X}_2$  found at Step 1, and, moreover,

$$\text{Aut}(\mathcal{X}) \leq \text{Aut}(\mathcal{X}_1) \times \text{Aut}(\mathcal{X}_2)$$

where the group in the right-hand side has odd order. Thus the correctness in this case follows from Theorem 10.1. Let  $\mathcal{X}$  be homogeneous. Then it is schurian only if for all  $\Gamma$  and  $\Gamma'$  defined at Step 2 we have

$$\mathcal{X}_\Gamma = \text{Inv}(\text{Aut}(\mathcal{X})^\Gamma) \quad \text{and} \quad \text{Aut}(\mathcal{X})_{\Gamma \rightarrow \Gamma'} \subset \text{Iso}(\mathcal{X}_\Gamma, \mathcal{X}_{\Gamma'}, \varphi_{\Gamma, \Gamma'})$$

where  $\text{Aut}(\mathcal{X})_{\Gamma \rightarrow \Gamma'}$  is the set of all bijections from  $\Gamma$  onto  $\Gamma'$  induced by the automorphisms of  $\mathcal{X}$ . On the other hand, the maximality condition in choosing  $e_i$ 's implies that under the schurity assumption each scheme  $\mathcal{X}_\Gamma$  is also primitive. Therefore by Theorem 1.1 in our case  $b(\mathcal{X}_\Gamma) \leq 3$  for all  $\Gamma$ . Thus the relations (55) imply that  $\mathcal{X}$  is not schurian, and the output of Step 3 is correct. Now, the correctness of the output at Step 4 follows from Lemma 10.2. Finally a polynomial bound for the running time of the algorithm follows from Theorem 10.1 and Corollary 3.6. ■

**10.4. Proof of Theorem 1.4.** Given a colored tournament  $T$  denote by  $\mathcal{X} = \mathcal{X}(T)$  the coherent configuration  $\mathcal{X}(T) = \text{Fis}(\mathcal{S})$  where  $\mathcal{S}$  is the set of color classes of the arc set of  $T$ . Then  $T$  is schurian if and only if the coherent configuration  $\mathcal{X}$  is schurian. Since the latter can be constructed in time  $n^{O(1)}$  where  $n$  is the number of vertices of  $T$  (see Subsection 2.8), statements (1) and (2) immediately follow from Theorem 1.3.

To prove statement (3) let  $T_i$  be a colored schurian tournament,  $\mathcal{S}_i$  the set of color classes of the arc set of  $T_i$  and  $\mathcal{X}_i = \mathcal{X}(T_i)$ ,  $i = 1, 2$ . Then by Theorem 2.4 without loss of generality we can assume that there exists an algebraic isomorphism  $\varphi : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  such that

$$(56) \quad \mathcal{S}_1^\varphi = \mathcal{S}_2$$

(for otherwise  $\text{Iso}(T_1, T_2) = \emptyset$ ). In this case  $\text{Iso}(T_1, T_2) = \text{Iso}(\mathcal{X}_1, \mathcal{X}_2, \varphi)$ . To construct the latter set take a copy  $\mathcal{X}_3$  of the coherent configuration  $\mathcal{X}_2$ . Set

$$\mathcal{W} = \{\mathcal{X}_i\}_{i=1}^3 \quad \text{and} \quad \Psi = \{\psi_{i,j}\}_{i,j=1}^3$$

where  $\psi_{i,j} : \mathcal{X}_i \rightarrow \mathcal{X}_j$  is an algebraic isomorphism defined as follows (below  $S_i$  denotes the set of basis relations of  $\mathcal{X}_i$ ):

- $\psi_{i,j} = \text{id}_{S_i}$ , if  $i = j$  or  $\{i, j\} = \{2, 3\}$ ,
- $\psi_{i,j} = \varphi$ , if  $1 = i \neq j$ ,
- $\psi_{i,j} = \varphi^{-1}$ , if  $i \neq j = 1$ .

According to [6, Definition 7.1] there exists the smallest coherent configuration  $\mathcal{X}$  on the disjoint union  $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$  where  $\Omega_i$  is the point set of  $\mathcal{X}_i$ , such that

$$\mathcal{X}_{\Omega_i} = \mathcal{X}_i \quad \text{and} \quad \mathcal{X}_{\Omega/e} = \text{Inv}(G)$$

where  $i = 1, 2, 3$ ,  $e = \Omega_1^2 \cup \Omega_2^2 \cup \Omega_3^2$  and  $G$  is the cyclic subgroup of  $\text{Sym}(3)$ . It was also proved there (see [6, Corollary 7.9]) that  $\mathcal{X}$  is schurian if and only if  $\mathcal{X}_i$  is schurian for all  $i$  and the algebraic isomorphism  $\psi_{i,j}$  is induced by an isomorphism for all  $i, j$ . In our case the former condition is obviously satisfied, whereas the latter one is satisfied if and only if the set  $\text{Iso}(\mathcal{X}_1, \mathcal{X}_2, \varphi)$  is not empty.

To complete the proof we note that the coherent configuration  $\mathcal{X}$  is antisymmetric. Therefore by Theorem 1.3 one can test whether or not  $\mathcal{X}$  is schurian and (if so) find the group  $\text{Aut}(\mathcal{X})$  in time  $n^{O(1)}$ . Now, if  $\mathcal{X}$  is not schurian, then by the above

$$\text{Iso}(\mathcal{X}_1, \mathcal{X}_2, \varphi) = \emptyset.$$

On the other hand, if  $\mathcal{X}$  is schurian, then by means of standard permutation group algorithms (see e.g. [20]) one can efficiently find an element  $g \in \text{Aut}(\mathcal{X})$  such that  $\Omega_1^g = \Omega_2$ , and the setwise stabilizer  $H$  of the set  $\Omega_1$  in the group  $\text{Aut}(\mathcal{X})$ . Since in this case obviously

$$\text{Iso}(\mathcal{X}_1, \mathcal{X}_2, \varphi) = \{h^{\Omega_1} g_{\Omega_1} : h \in H\}$$

where  $h^{\Omega_1}$  is the restriction of  $h$  on  $\Omega_1$ , and  $g_{\Omega_1} : \Omega_1 \rightarrow \Omega_2$  is the bijection induced by  $g$ , we are done.  $\blacksquare$

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